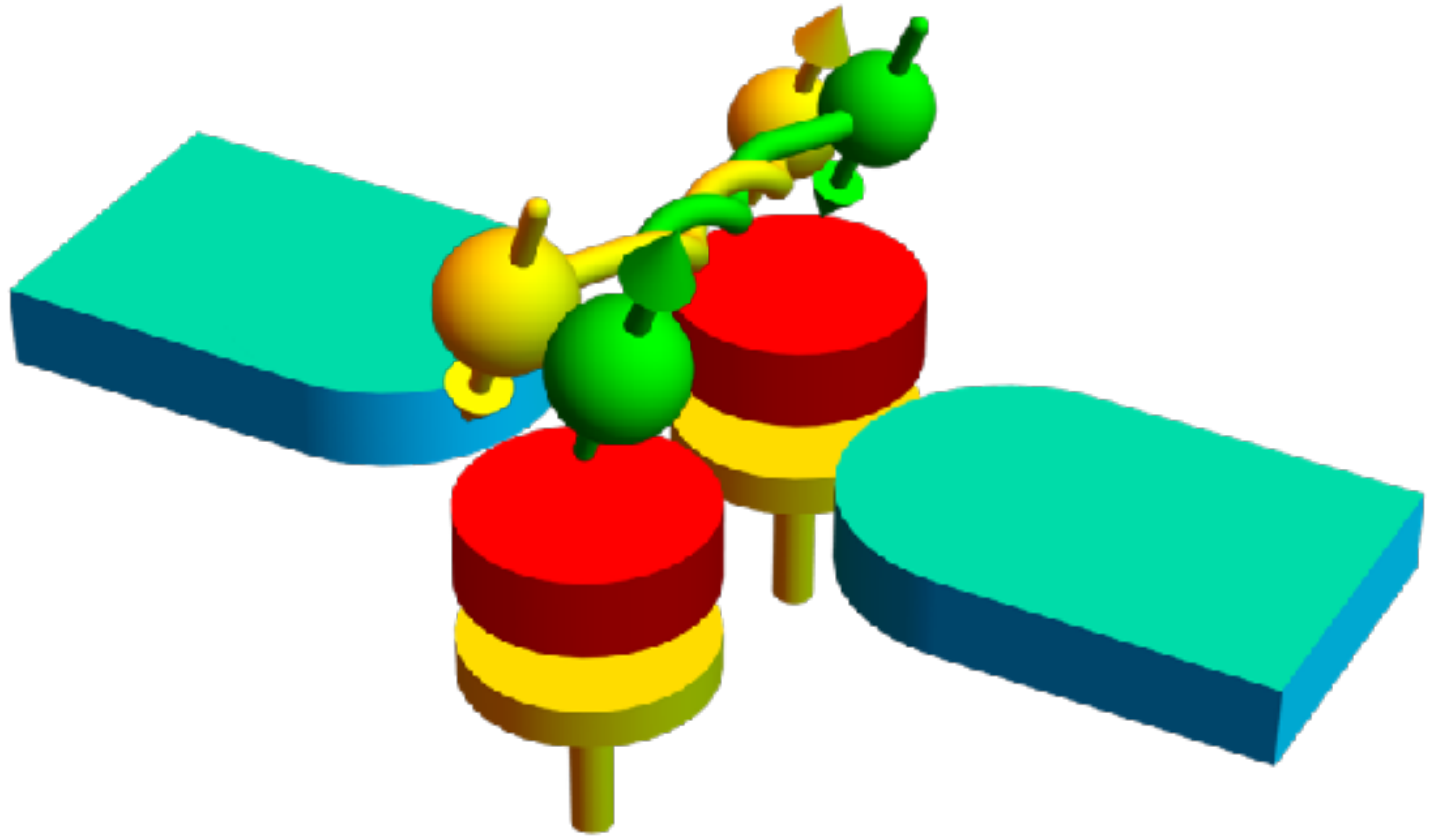
PQChips 2026 (Gdańsk, August 28, 2026)

ALGEBRAIC QUANTUM ALGORITHMS

MAHN-SOO CHOI (KOREA UNIVERSITY)

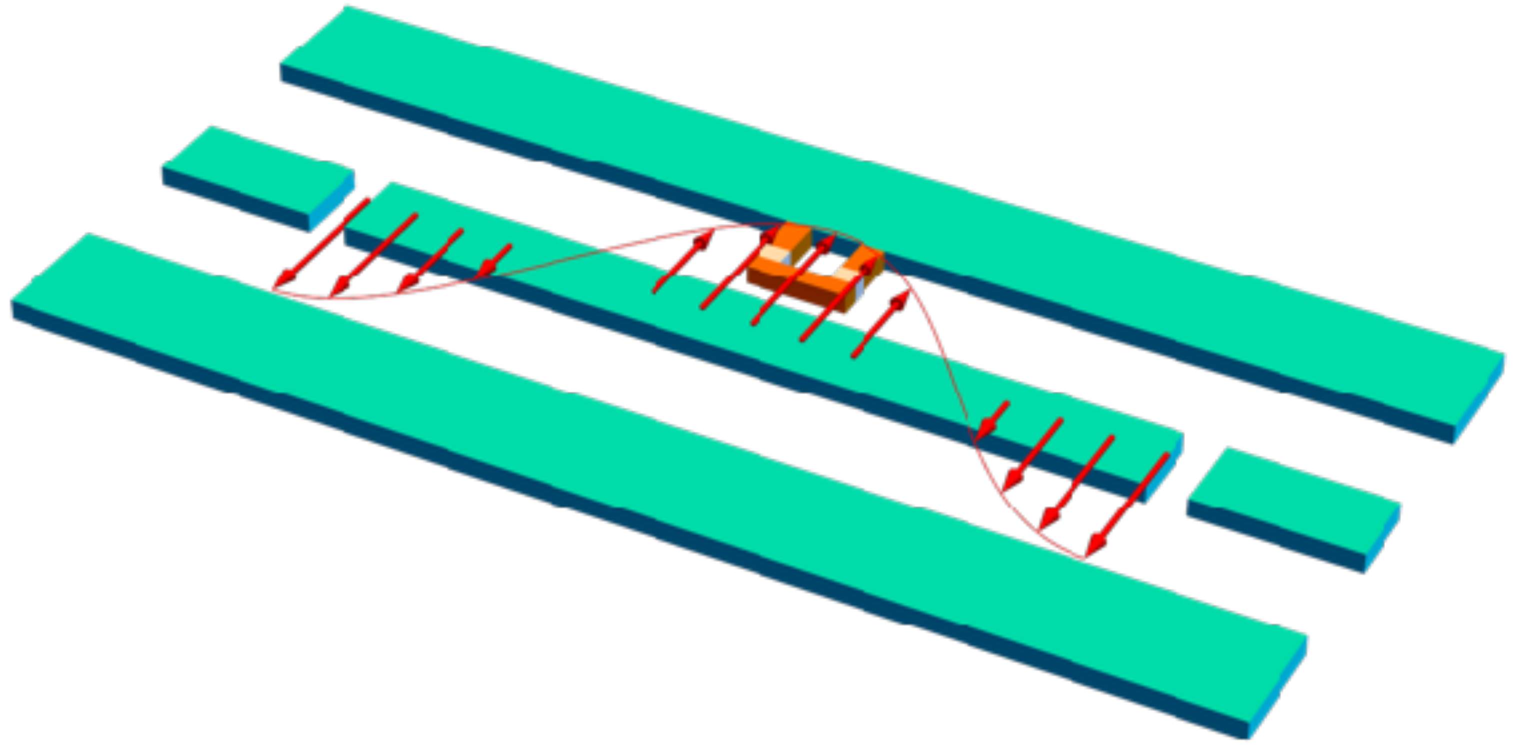
SCHOOL OF QUANTUM
KOREA UNIVERSITY HQ

spin entanglement
via superconductors



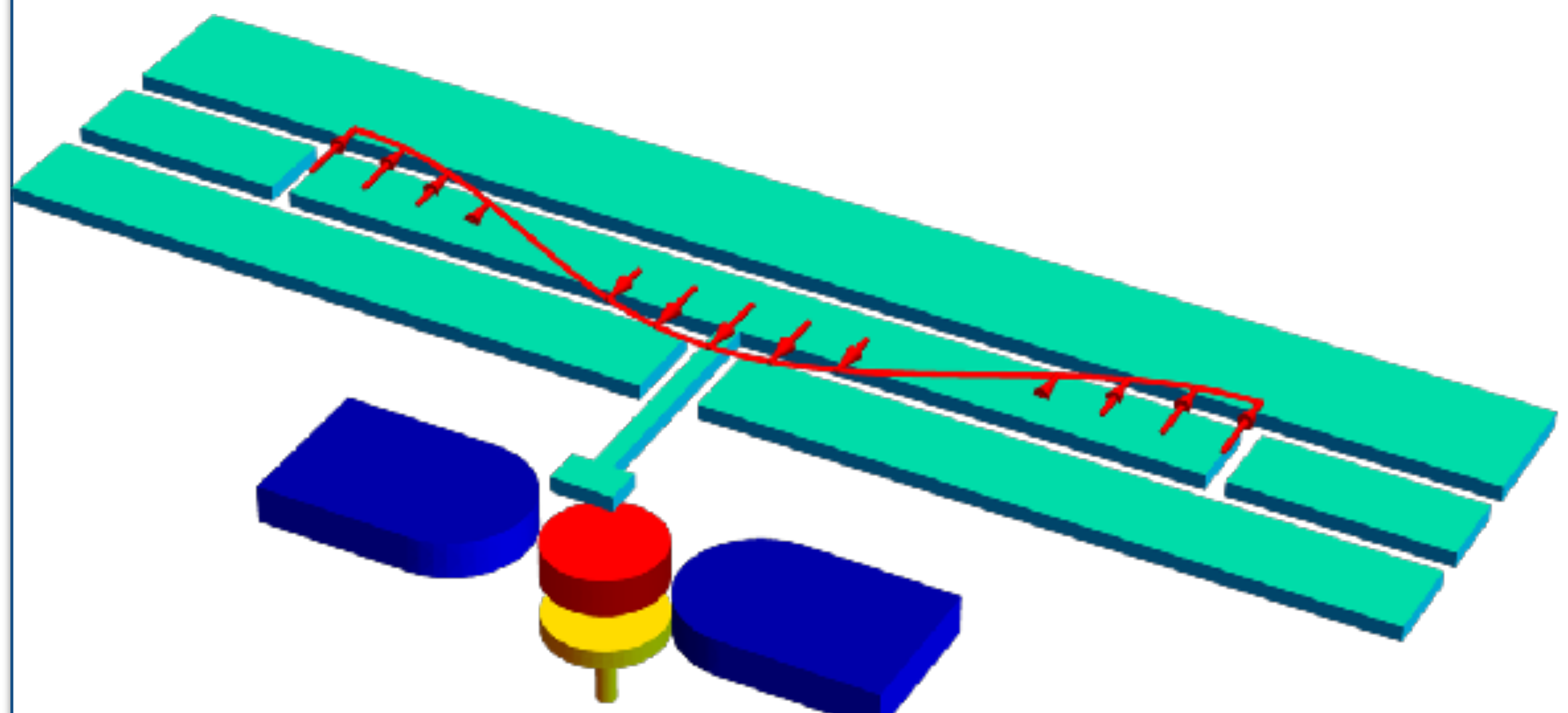
Choi, Bruder, Loss (PRB, 2000)
Choi, Sanchez, Lopez (PRL, 2004)
Choi, Lee, Kang, Belzig (PRL, 2005)

circuit QED
& strong-coupling physics



Choi (Adv Quant Tech, 2020)
Hwang, Kim, Choi (PRL, 2016)
Hwang, Choi (PRA, 2010)

quantum transport
in hybrid quantum circuits

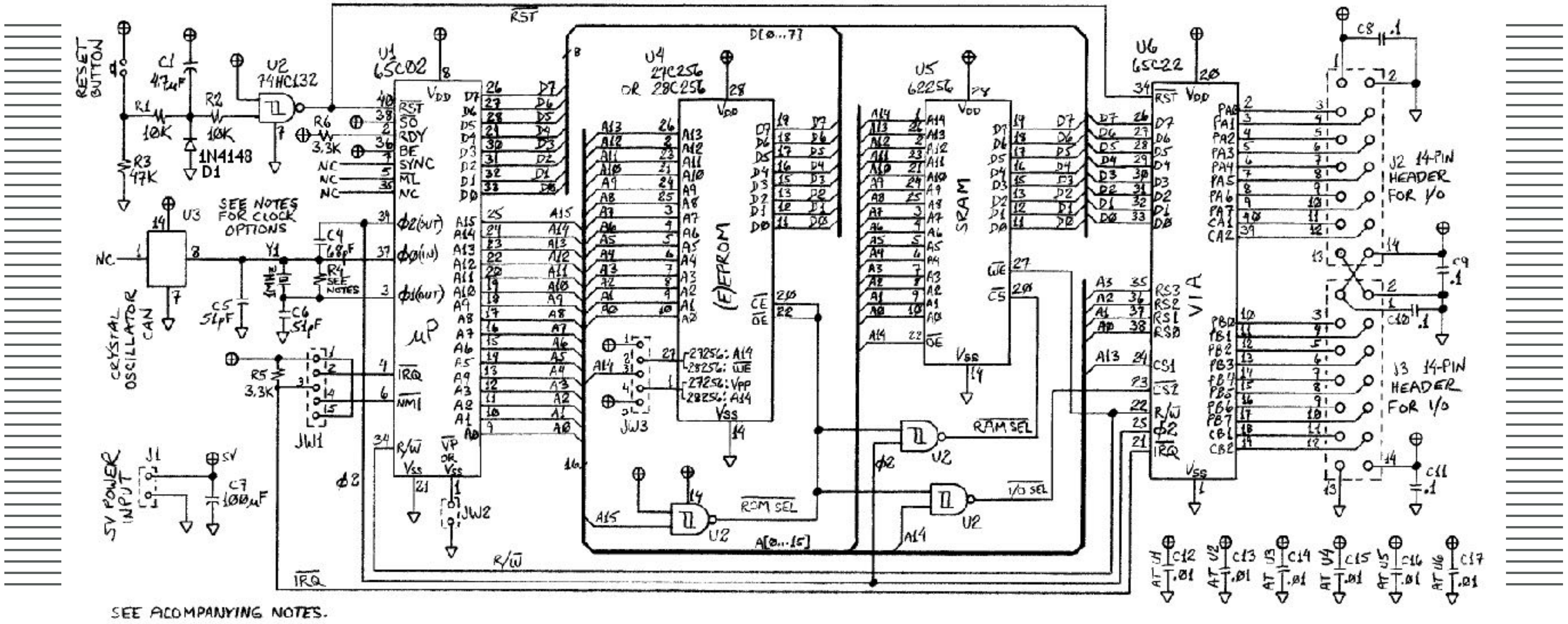


Desjardins *et al.* (Nature, 2017)
Delattre *et al.* (Nat Phys, 2009)

Why is it faster?
Or, is it at all?

Super-power vs Super-weakness

FACTORING 713



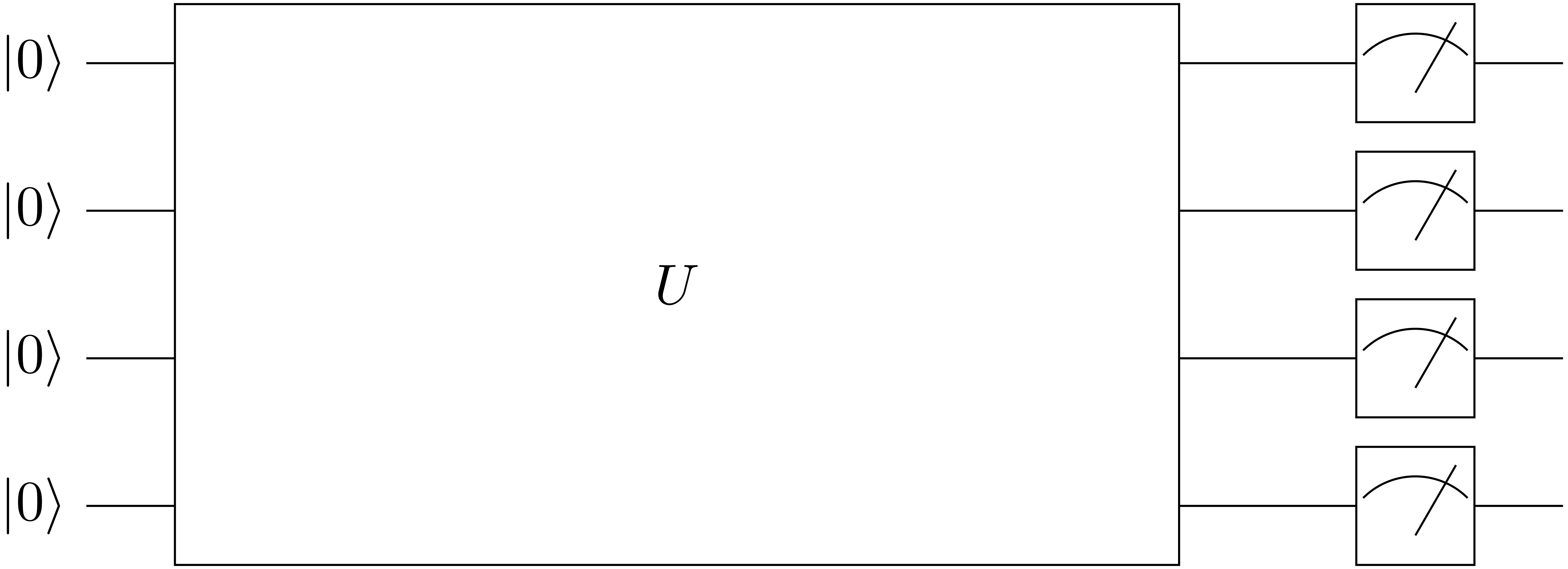
$713 = 23 \times 31$

~~1~~	~~2~~	~~3~~	~~4~~	~~5~~	~~6~~	~~7~~	~~8~~	~~9~~	~~10~~	~~11~~	~~12~~
~~13~~	~~14~~	~~15~~	~~16~~	~~17~~	~~18~~	~~19~~	~~20~~	~~21~~	~~22~~	23	~~24~~
~~25~~	~~26~~	~~27~~	~~28~~	~~29~~	~~30~~	31	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~
37	38	39	40	41	42	43	44	45	46	47	48

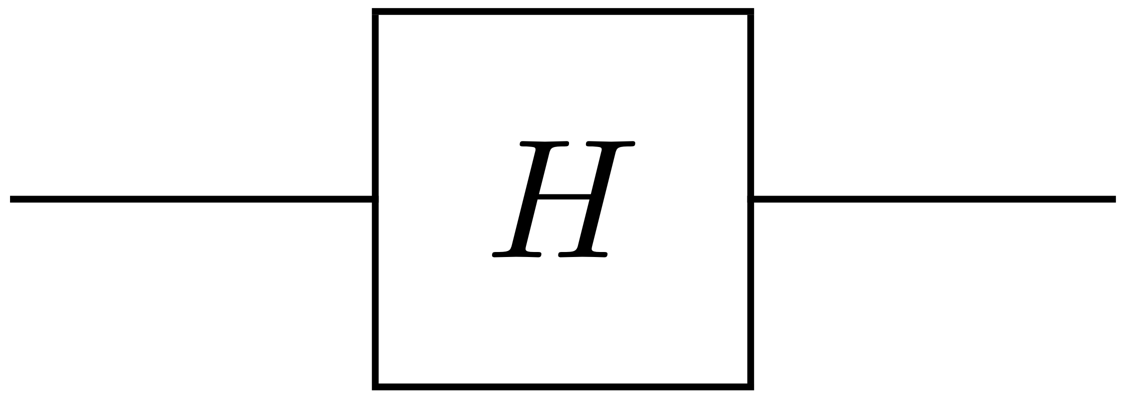
Quantum Computing

Overview

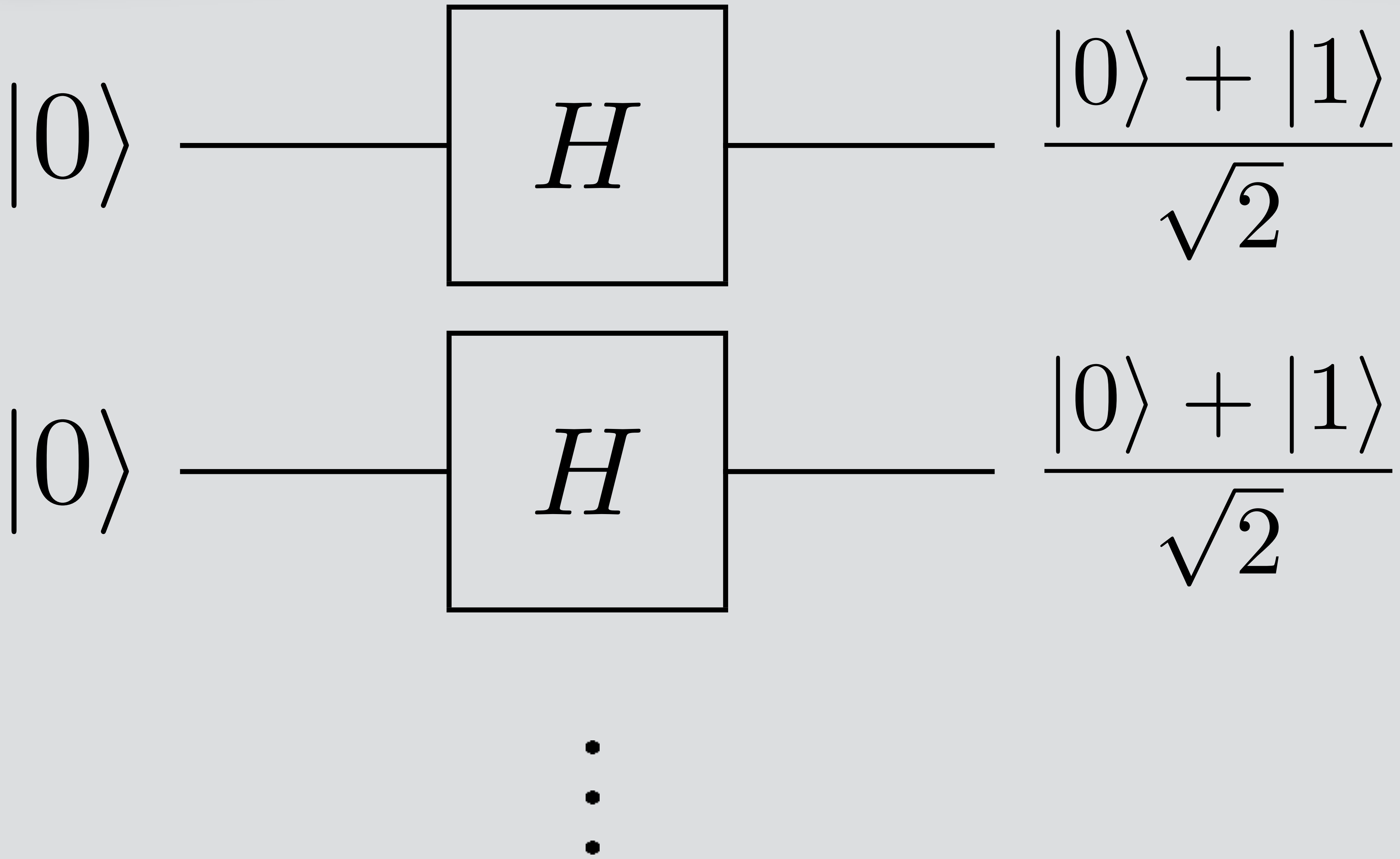
1. Initialization
2. Unitary operation
3. Measurement



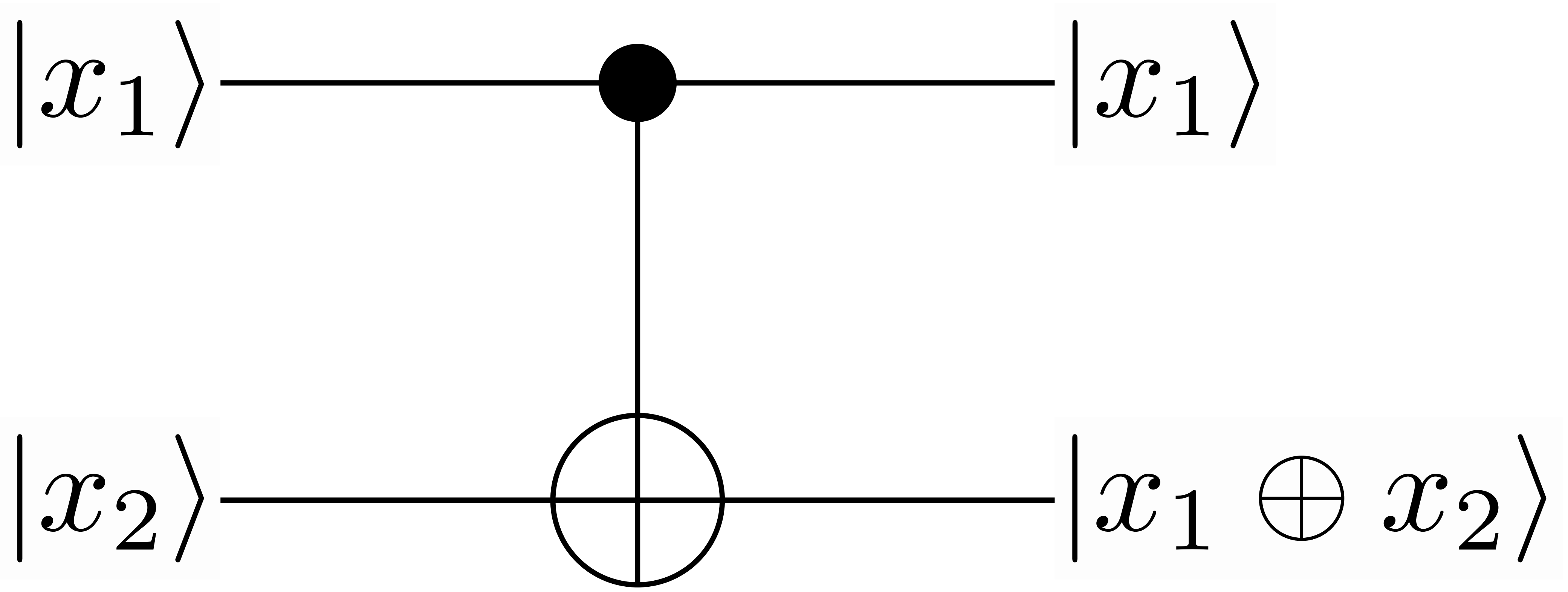
1
initialization
2
make superposition
$x = (x_1 x_2 \cdots x_n)_2$
4
remove superposition
measurement
5
$|0\rangle$
$|0\rangle$
$|0\rangle$
$|\phi\rangle$
H
H
H
x_1
x_2
x_3
x
U^x
QFT
$|y_1\rangle$
$|y_2\rangle$
$|y_3\rangle$
3
main calculation

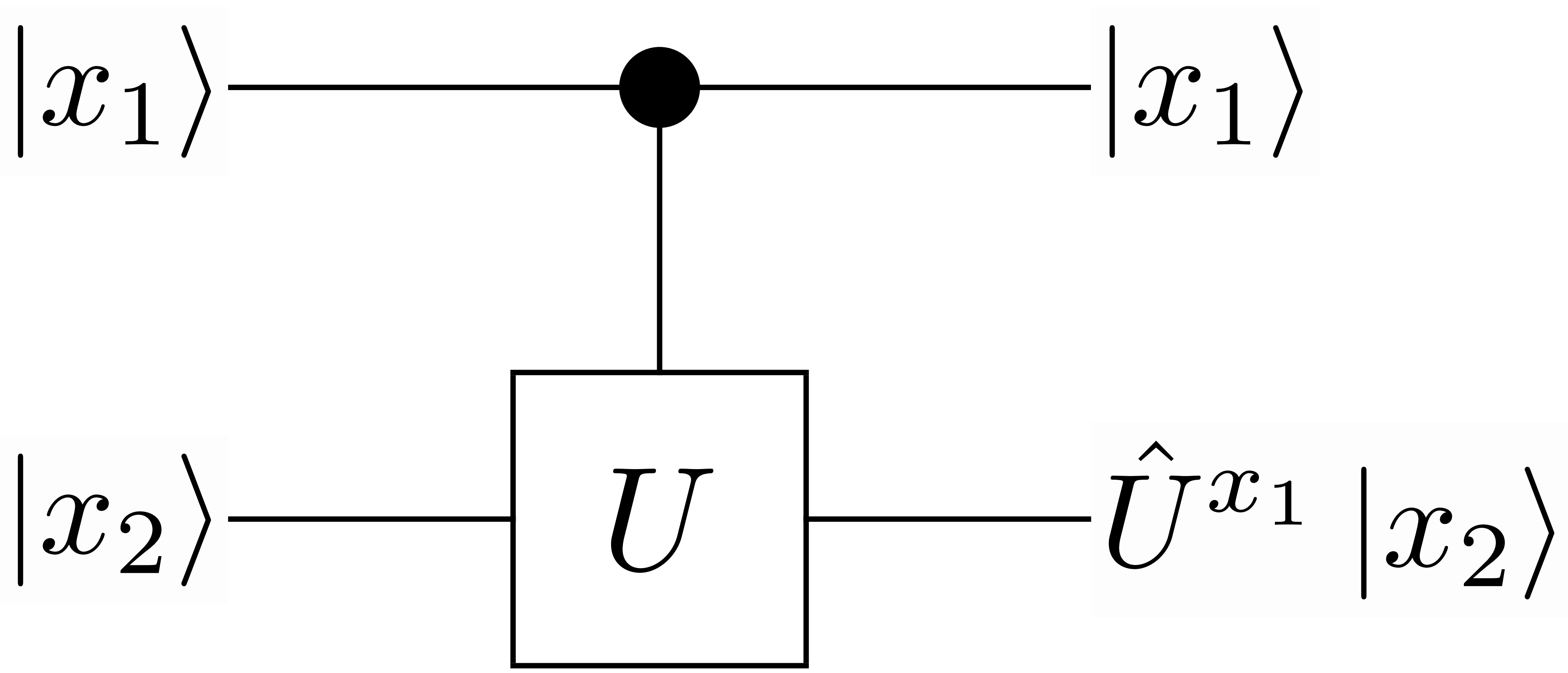


$$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$$



$$\begin{aligned}(|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle) \otimes \cdots \\ &= |00\cdots\rangle + |01\cdots\rangle + |10\cdots\rangle + |11\cdots\rangle + \cdots \\ &= |0\rangle + |1\rangle + |2\rangle + \cdots + |2^n - 1\rangle\end{aligned}$$

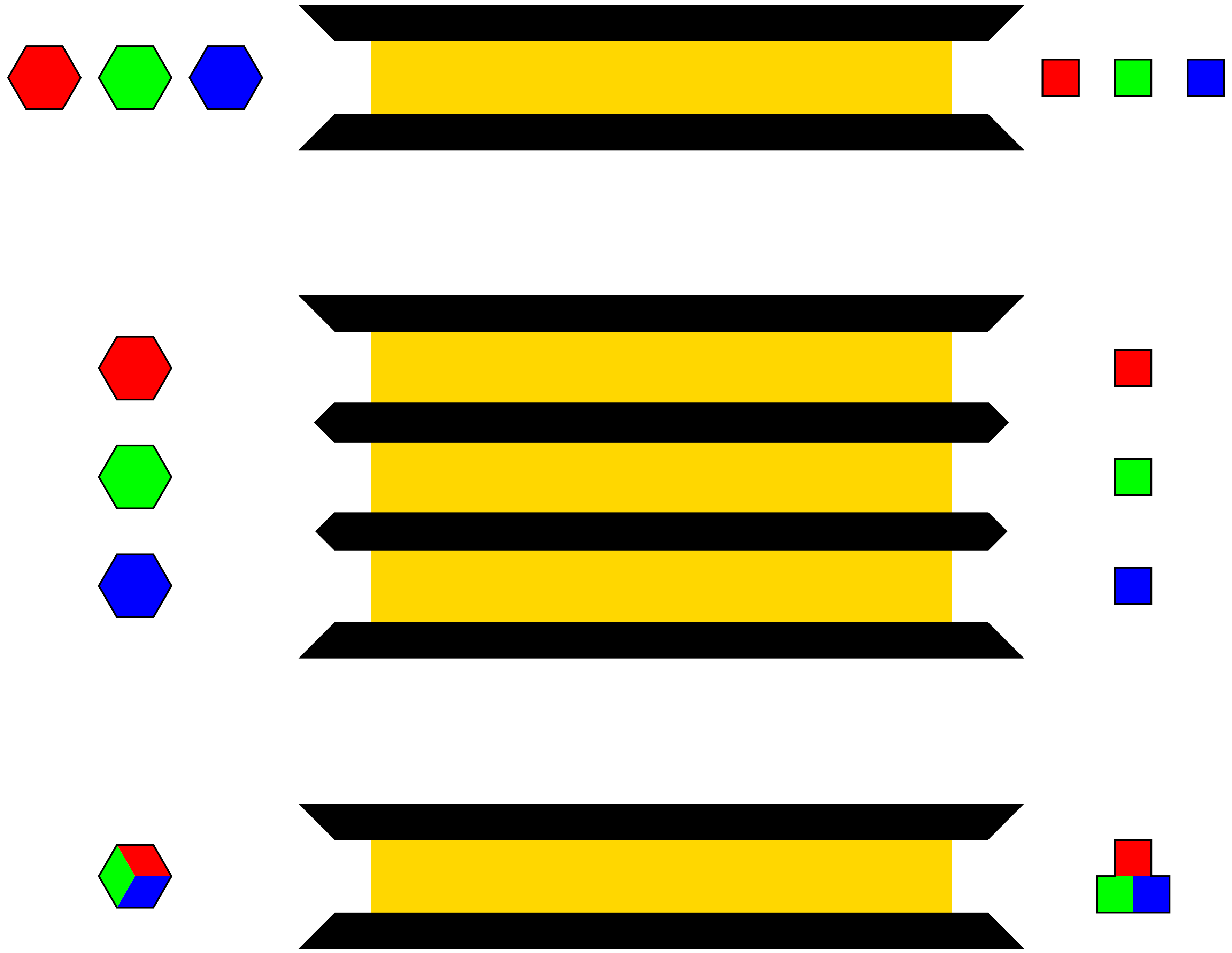




Factoring 713

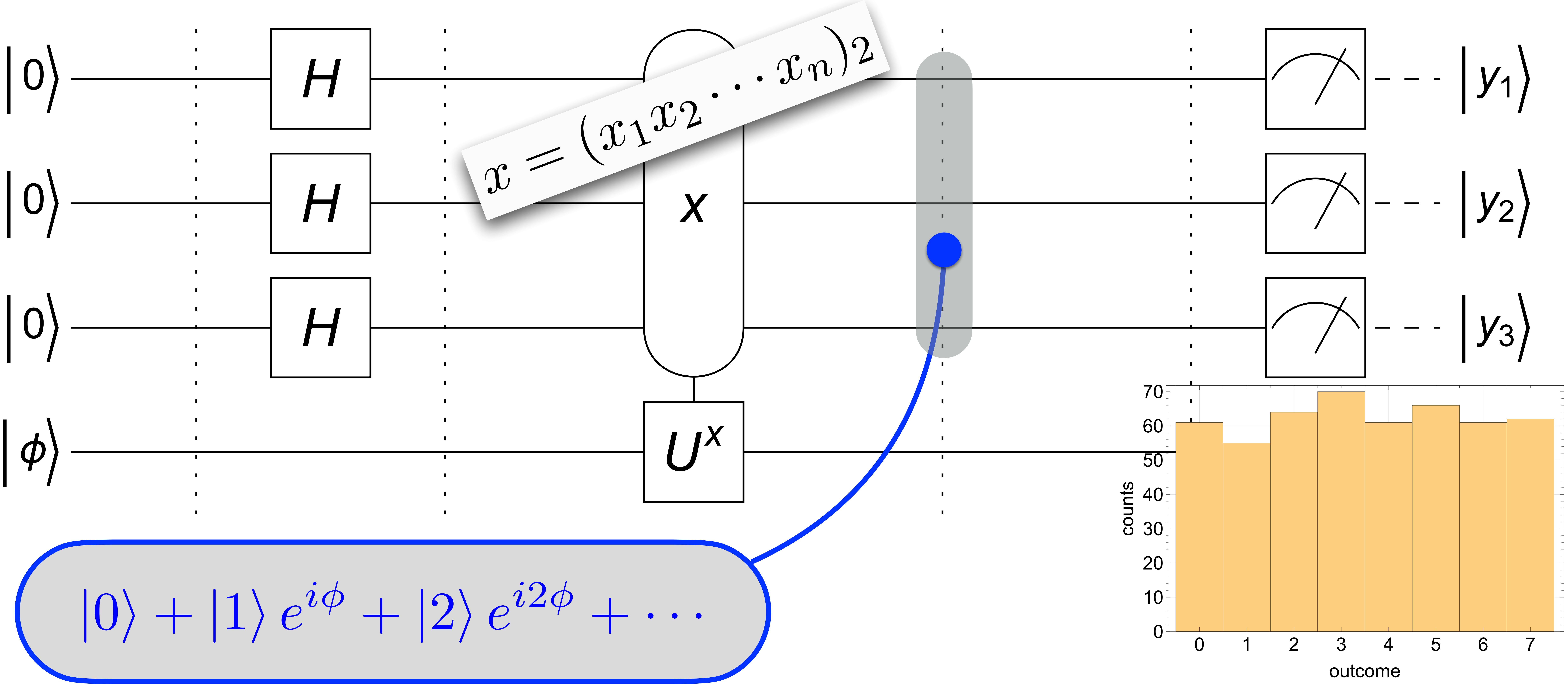
Quantum Super-Power

Quantum Parallelism

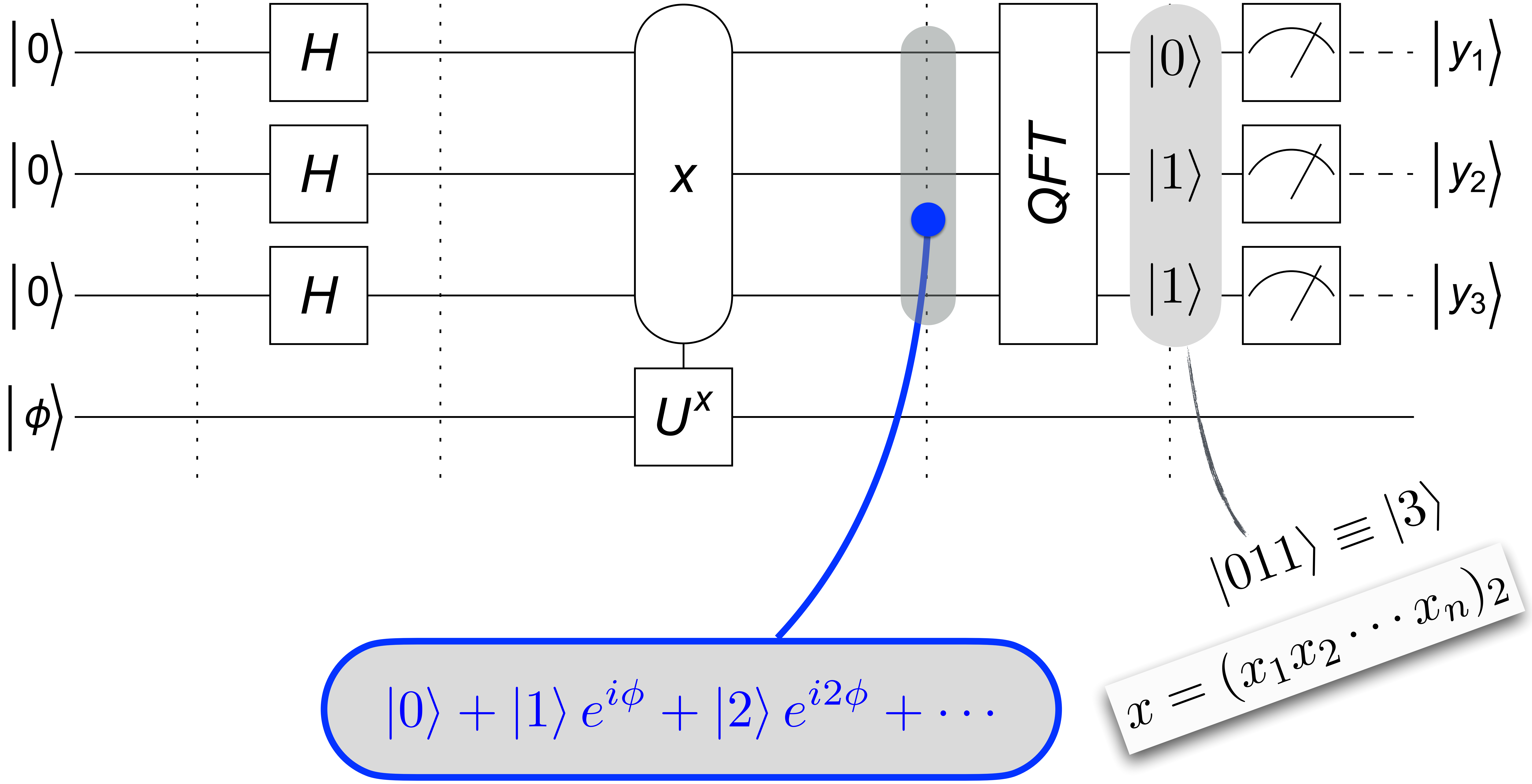


$$f(a\,|0\rangle + b\,|1\rangle) = a\,|f(0)\rangle + b\,|f(1)\rangle$$

Why not just measure?



Curse of complementarity!

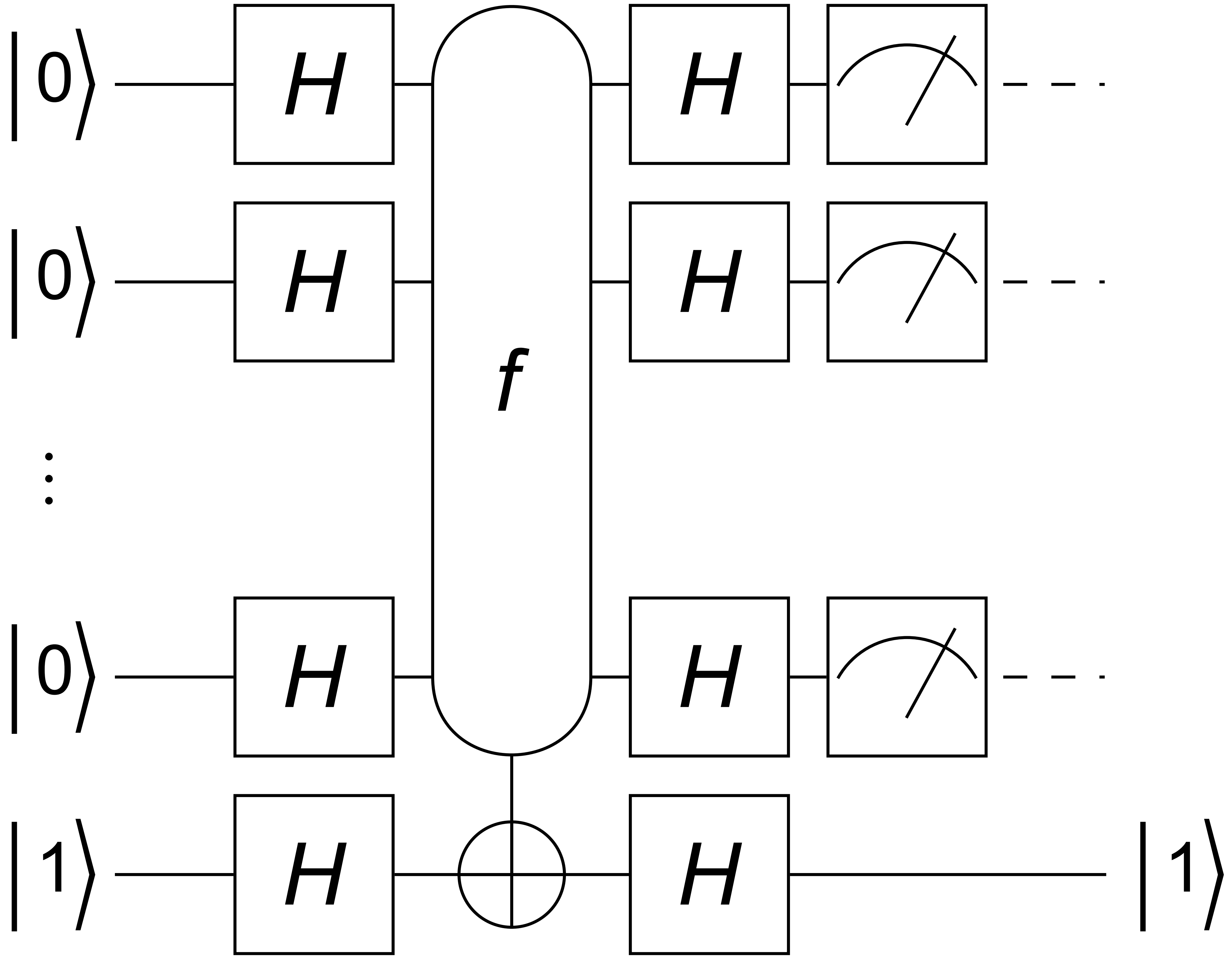


$|0\rangle$
$|0\rangle$
$|0\rangle$
$|\phi\rangle$
H
H
H
x
U^x
QFT
$|0\rangle$
$|1\rangle$
$|1\rangle$
$|y_1\rangle$
$|y_2\rangle$
$|y_3\rangle$
$|011\rangle \equiv |3\rangle$
$x = (x_1 x_2 \dots x_n)_2$
$|0\rangle + |1\rangle e^{i\phi} + |2\rangle e^{i2\phi} + \cdots$


Incident wavefront
Huygens' waveletts
P
Optical axis
θ
d
$d \sin\theta$
Focal plane
Diffraction grating
Lens
QFT
$|0\rangle$
$|1\rangle$
$|1\rangle$
$|y_1\rangle$
$|y_2\rangle$
$|y_3\rangle$
$|011\rangle \equiv |3\rangle$
$x = (x_1 x_2 \cdots x_n)_2$
$|0\rangle + |1\rangle e^{i\phi} + |2\rangle e^{i2\phi} + \cdots$

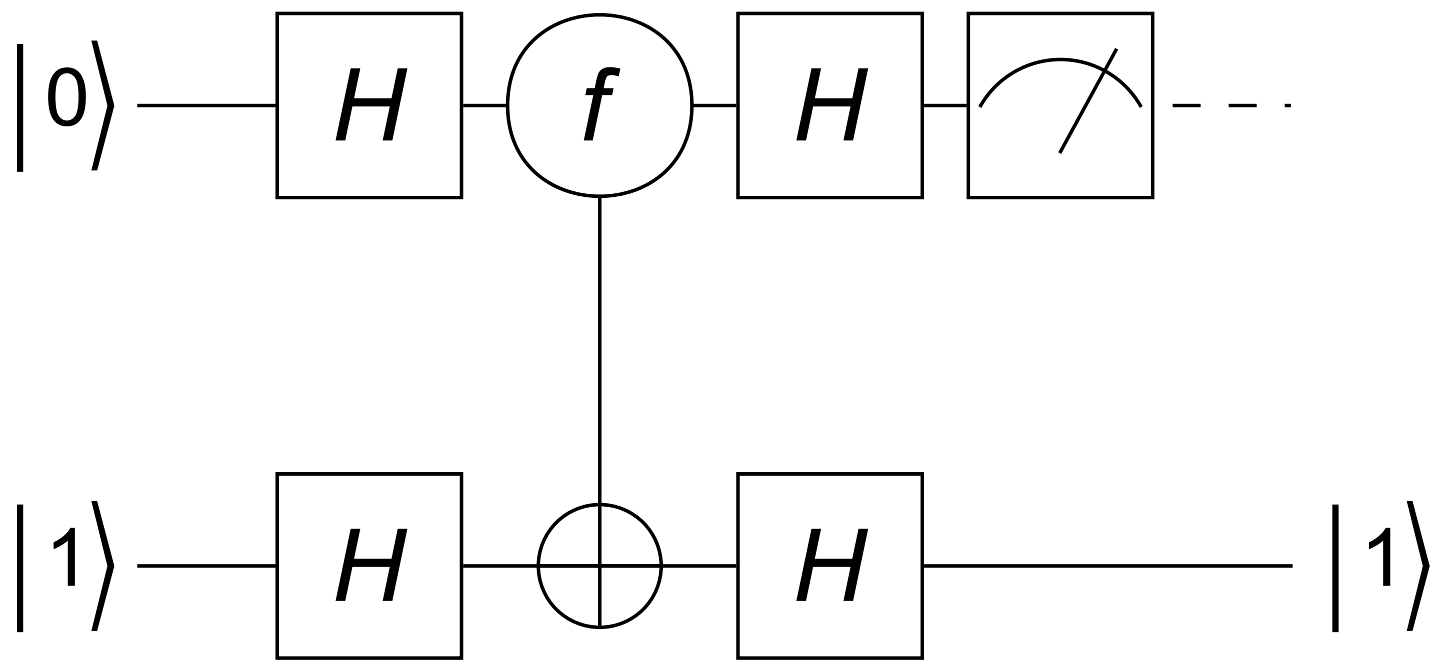
Deutsch-Jozsa Problem

Quantum Algorithm



Deutsch-Jozsa Problem

Quantum Algorithm



$$|0\rangle \otimes |1\rangle \xrightarrow{\hat{H}\otimes\hat{H}} \frac{1}{2}(|0\rangle + |1\rangle) \otimes (|0\rangle - |1\rangle)$$

$$= \frac{1}{2}\left\{|0\rangle \otimes (|0\rangle - |1\rangle) + |1\rangle \otimes (|0\rangle - |1\rangle)\right\}$$

$$\xrightarrow{\text{Oracle}} \frac{1}{2}\left\{|0\rangle \otimes (|0\rangle - |1\rangle)(-1)^{f(0)} + |1\rangle \otimes (|0\rangle - |1\rangle)(-1)^{f(1)}\right\}$$

$$= \frac{1}{2}\left\{|0\rangle (-1)^{f(0)} + |1\rangle (-1)^{f(1)}\right\} \otimes (|0\rangle - |1\rangle)$$

$$= \begin{cases} (|0\rangle + |1\rangle) \otimes (|0\rangle - |1\rangle), & \text{(constant)} . \\ (|0\rangle - |1\rangle) \otimes (|0\rangle - |1\rangle), & \text{(balanced)}. \end{cases}$$

1
initialization
2
make superposition
4
remove superposition
measurement
5
|0⟩
|0⟩
|0⟩
|φ⟩
H
H
H
x
U^x
NOT ALWAYS EASY!
|y1⟩
|y2⟩
|y3⟩
3
main calculation

Consequences of Linearity

Bless or Curse?

Duality
(quantum parallelism)

- Ability to try all possibilities at once

Super-power

Complementarity
(no-cloning theorem)

- Only orthogonal states may be discriminated/copied

Super-curse

Quantum Algorithms: Two Families

- Quantum Fourier Transform Based
  - Generalized Quantum Fourier Transform
  - Quantum Schur Transform

- Quantum Search Based
  - Quantum Amplitude Amplification
  - Quantum Signal Processing
  - Quantum Singular Value Transform

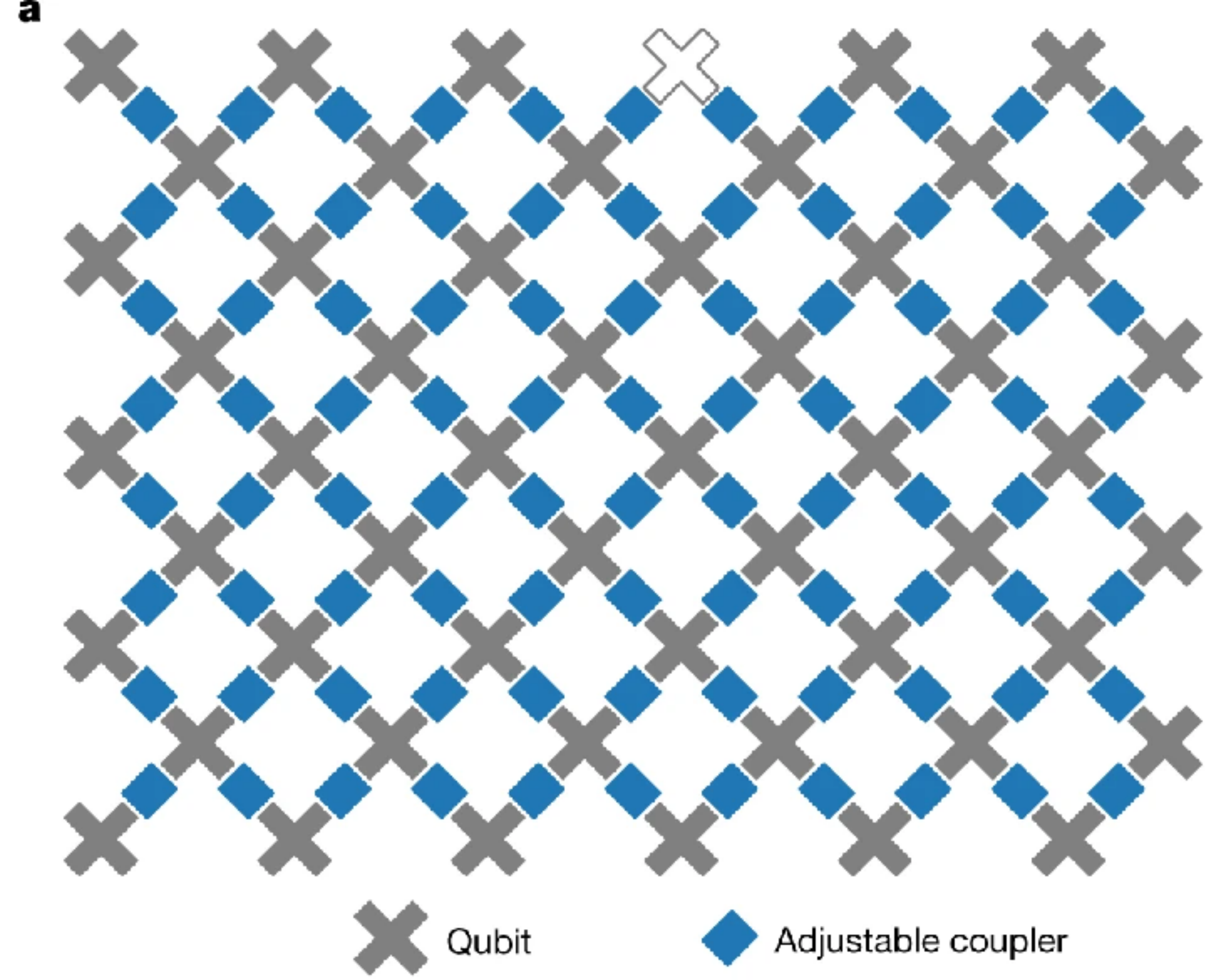
- Hybrid Types
  - VQA, QML, etc.

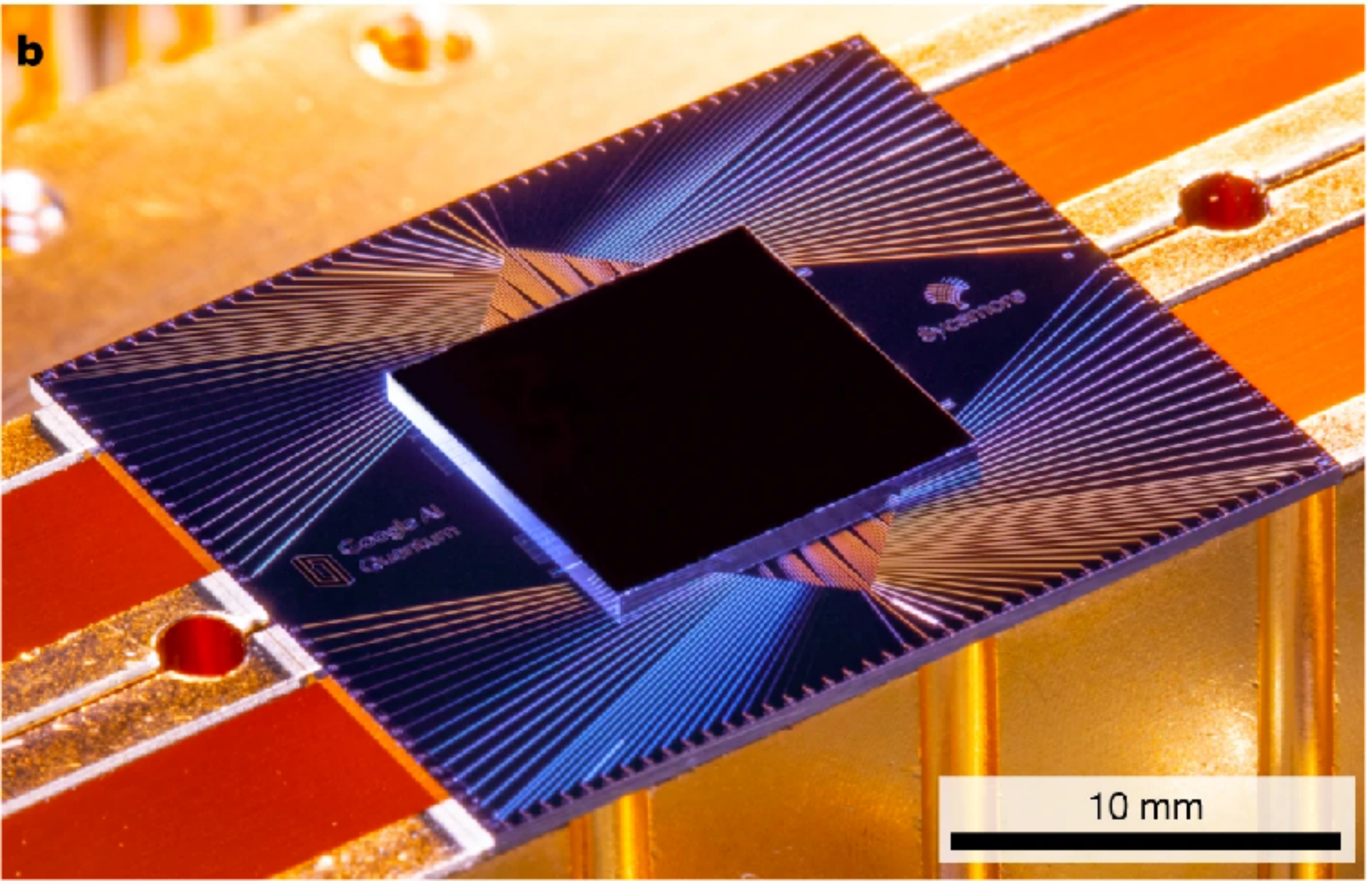
- cf. Decoded Quantum Interferometry (arXiv: 2408.08292)

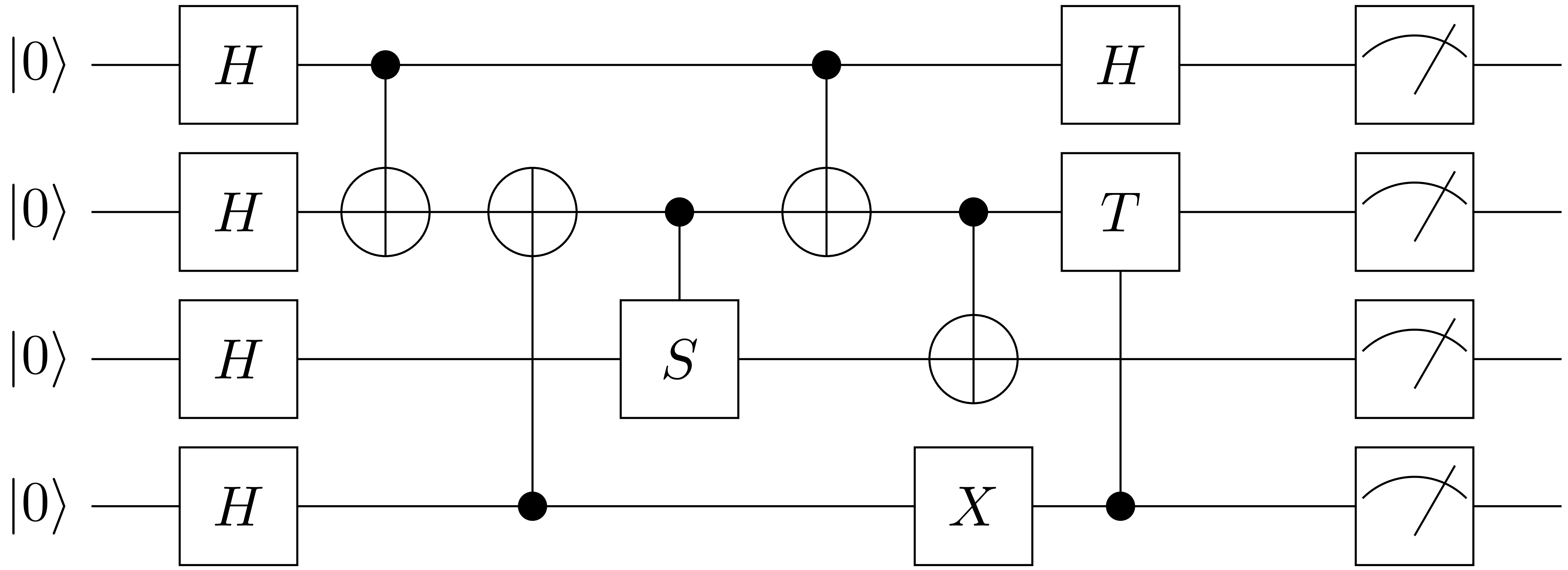
Sampling Problems

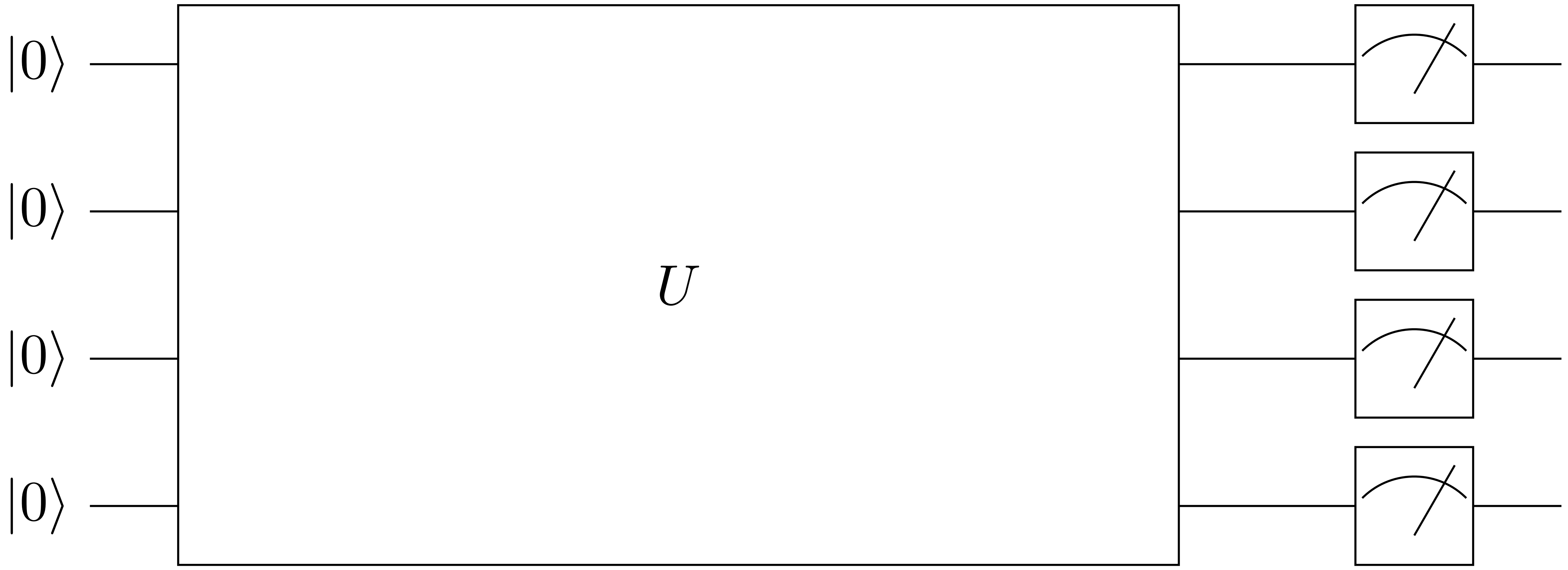
- Random Circuit Sampling
- Boson Sampling
- …

RANDOM CIRCUIT SAMPLING



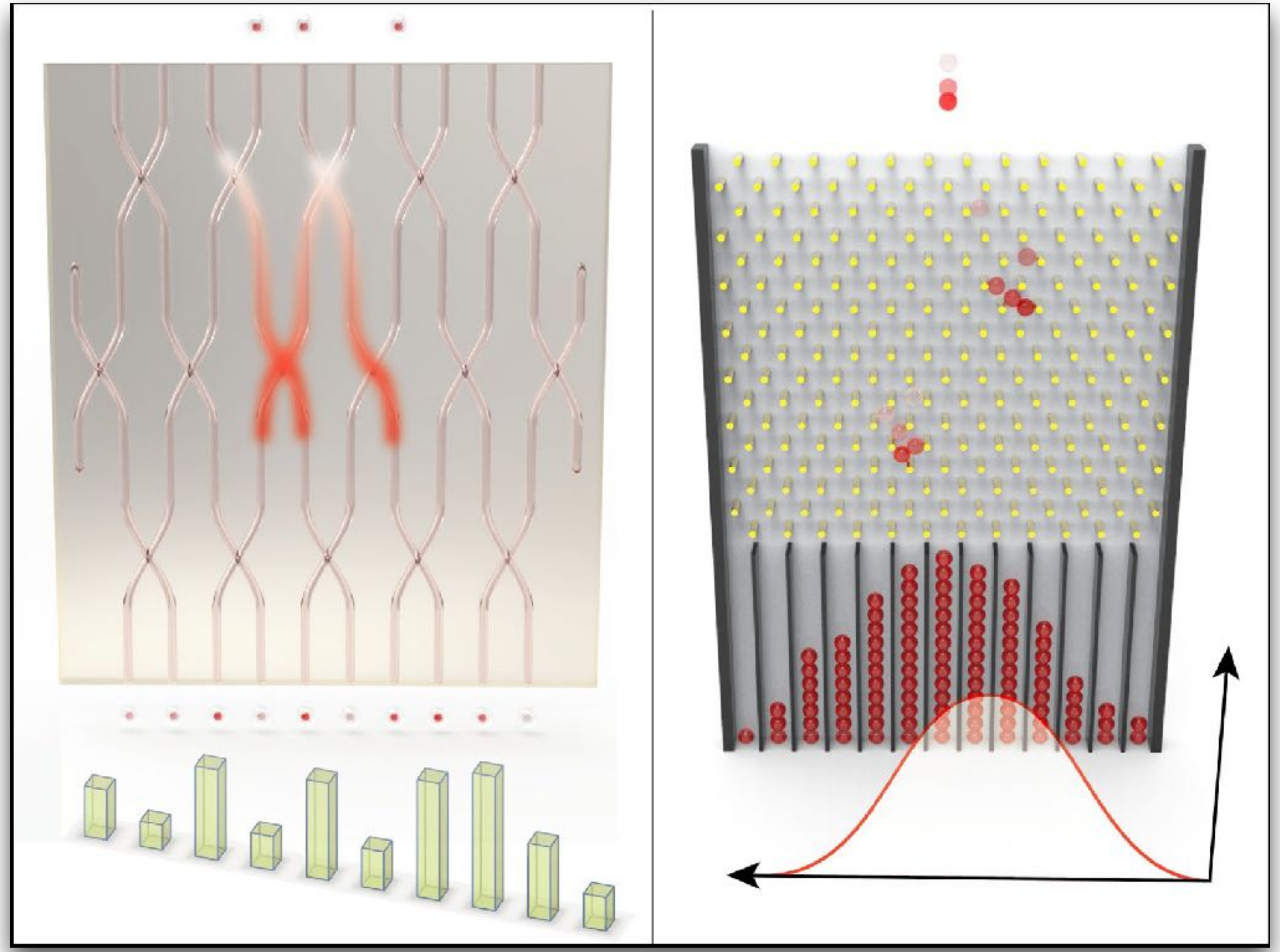


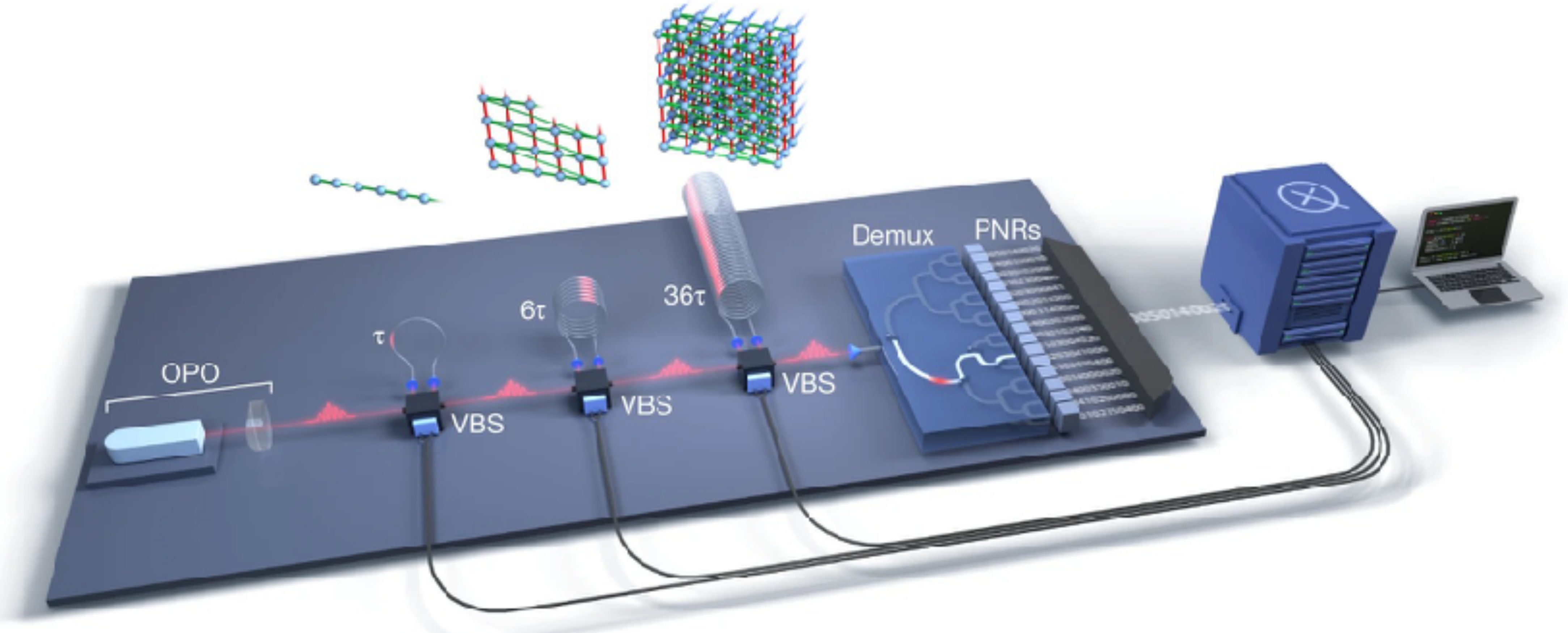




Arute *et al*. (Nature, 2019)

Boson Sampling





Madsen et al. (Nature, 2022)

…. certain families of constant-depth, 2D random circuits can be approximately simulated on a classical computer in time only linear in the number of qubits and gates, …

PHYSICAL REVIEW X **12,** 021021 (2022)

Efficient Classical Simulation of Random Shallow 2D Quantum Circuits

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A central question of quantum computing is determining the source of the advantage of quantum computation over classical computation. Even though simulating quantum dynamics on a classical computer is thought to require exponential overhead in the worst case, efficient simulations are known to exist in several special cases. It was widely assumed that these easy-to-simulate cases as well as any yet-undiscovered ones could be avoided by choosing a quantum circuit at random. We prove that this intuition is false by showing that certain families of constant-depth, 2D random circuits can be approximately simulated on a classical computer in time only linear in the number of qubits and gates, even though the same families are capable of universal quantum computation and are hard to exactly simulate in the worst case (under standard hardness assumptions). While our proof applies to specific random circuit families, we demonstrate numerically that typical instances of more general families of sufficiently shallow constant-depth 2D random circuits are also efficiently simulable. We propose two classical simulation algorithms. One is based on first simulating spatially local regions which are then "stitched" together via recovery maps. The other reduces the 2D simulation problem to a problem of simulating a form of 1D dynamics consisting of alternating rounds of random local unitaries and weak measurements. Similar processes have recently been the subject of an intensive research focus, which has observed that the dynamics generally undergo a phase transition from a low-entanglement (and efficient-to-simulate) regime to a high-entanglement (and inefficient-to-simulate) regime as measurement strength is varied. Via a mapping from random quantum circuits to classical statistical mechanical models, we give analytical evidence that a similar computational phase transition occurs for both of our algorithms as parameters of the circuit architecture like the local Hilbert space dimension and circuit depth are varied and, additionally, that the effective 1D dynamics corresponding to sufficiently shallow random quantum circuits falls within the efficient-to-simulate regime. Implementing the latter algorithm for the depth-3 "brickwork" architecture, for which exact simulation is hard, we find that a laptop could simulate typical instances on a 409×409 grid with a total variation distance error less than 0.01 in approximately one minute per sample, a task intractable for previously known circuit simulation algorithms. Numerical results support our analytic evidence that the algorithm is asymptotically efficient.

DOI: 10.1103/PhysRevX.12.021021 Subject Areas: Quantum Physics, Quantum Information, Statistical Physics

I. MOTIVATION

Simulating quantum systems on classical computers requires effort that scales exponentially with the size of the quantum system, at least for a general-purpose simulation. This was the original motivation of the field of quantum computing, since it shows that time evolution (natural or controlled) of quantum systems can perform tasks that are intractable for classical computers. However, many special cases are known where quantum time evolution *can* be efficiently simulated, including limited entanglement or interaction, free fermions, and Clifford circuits. Understanding the boundary between classically simulable and intractable gets at a crucial question: What makes quantum computing powerful?

There are two main ways to answer this question concretely. We can find more classes of easy-to-simulate quantum dynamics, or we can find evidence that other classes of quantum dynamics are hard for classical computers to simulate. The latter approach is related to the goal of *quantum computational supremacy* [1], which involves finding a well-defined computational task with evidence for classical intractability (usually based on a plausible

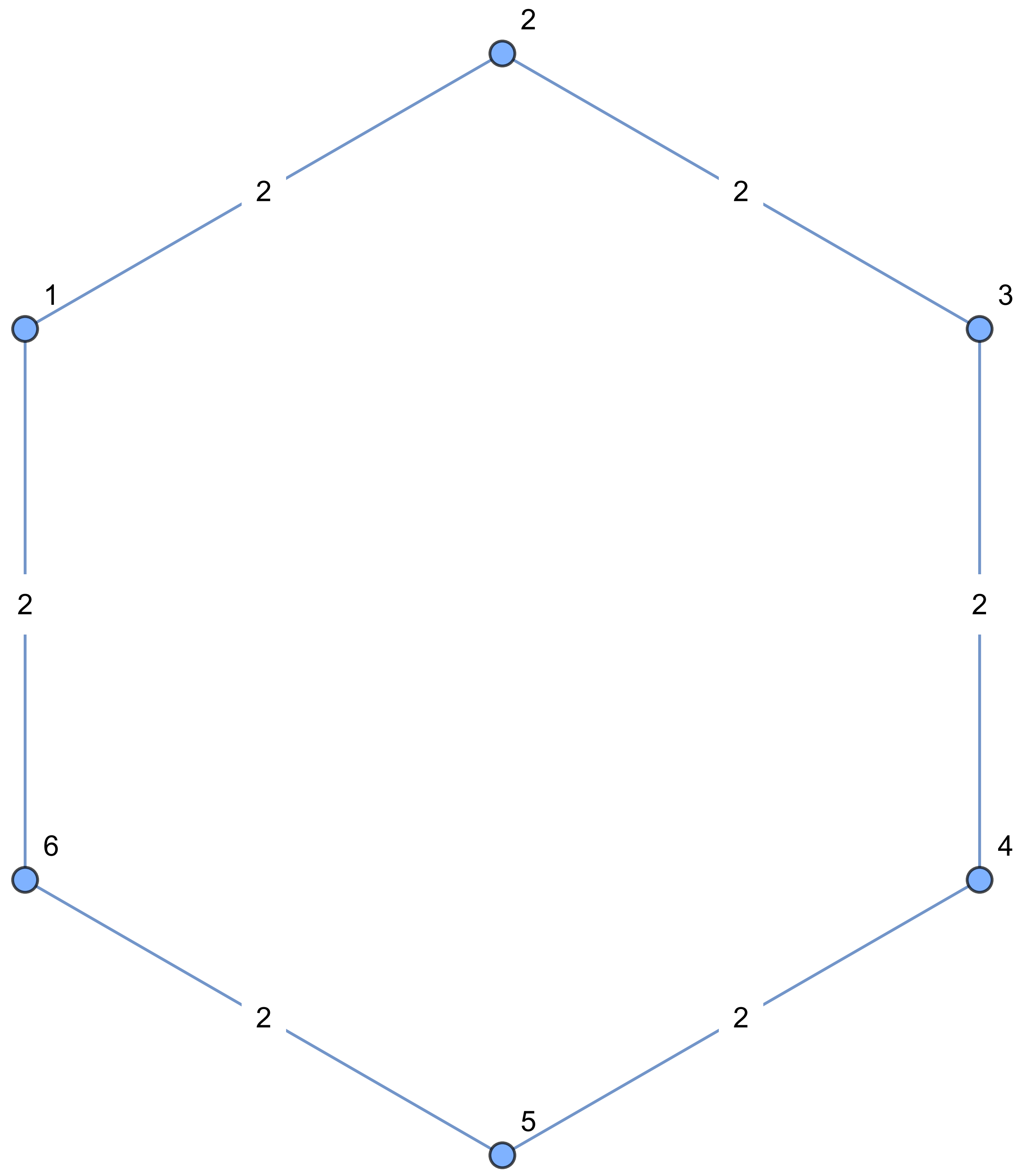
Quantum Fourier Transform

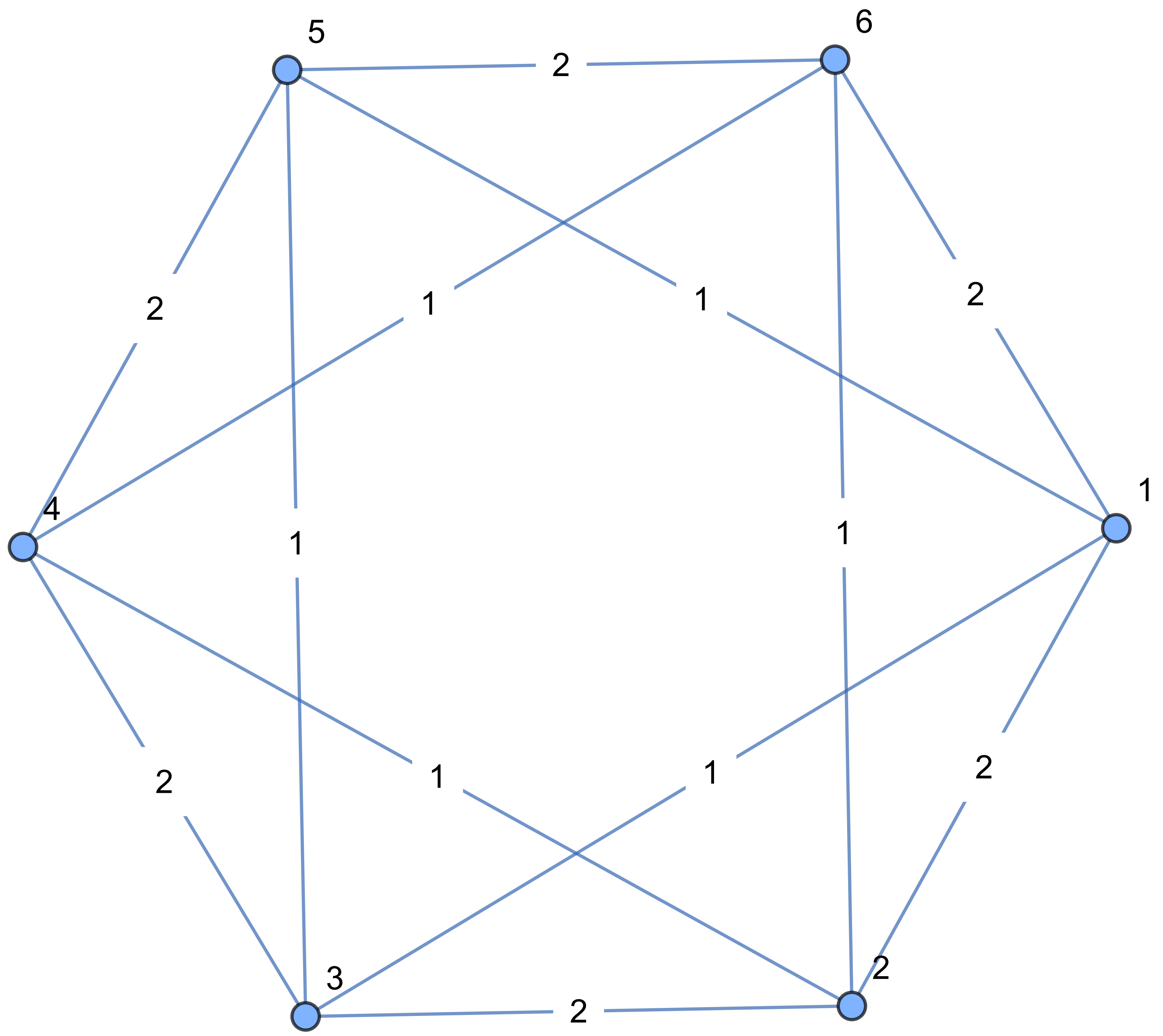
Can you diagonalize these?

$$\begin{pmatrix} 3 & 2 & 0 & 0 & 0 & 2 \\ 2 & 3 & 2 & 0 & 0 & 0 \\ 0 & 2 & 3 & 2 & 0 & 0 \\ 0 & 0 & 2 & 3 & 2 & 0 \\ 0 & 0 & 0 & 2 & 3 & 2 \\ 2 & 0 & 0 & 0 & 2 & 3 \end{pmatrix} \qquad \begin{pmatrix} 3 & 2 & 1 & 0 & 1 & 2 \\ 2 & 3 & 2 & 1 & 0 & 1 \\ 1 & 2 & 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 & 2 & 1 \\ 1 & 0 & 1 & 2 & 3 & 2 \\ 2 & 1 & 0 & 1 & 2 & 3 \end{pmatrix}$$

Translational Symmetry

(Abelian Symmetry)





They are diagonalized ….

$$
\begin{pmatrix}
7 & 0 & 0 & 0 & 0 & 0 \\
0 & 5 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 5
\end{pmatrix}
\qquad
\begin{pmatrix}
9 & 0 & 0 & 0 & 0 & 0 \\
0 & 4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 4
\end{pmatrix}
$$

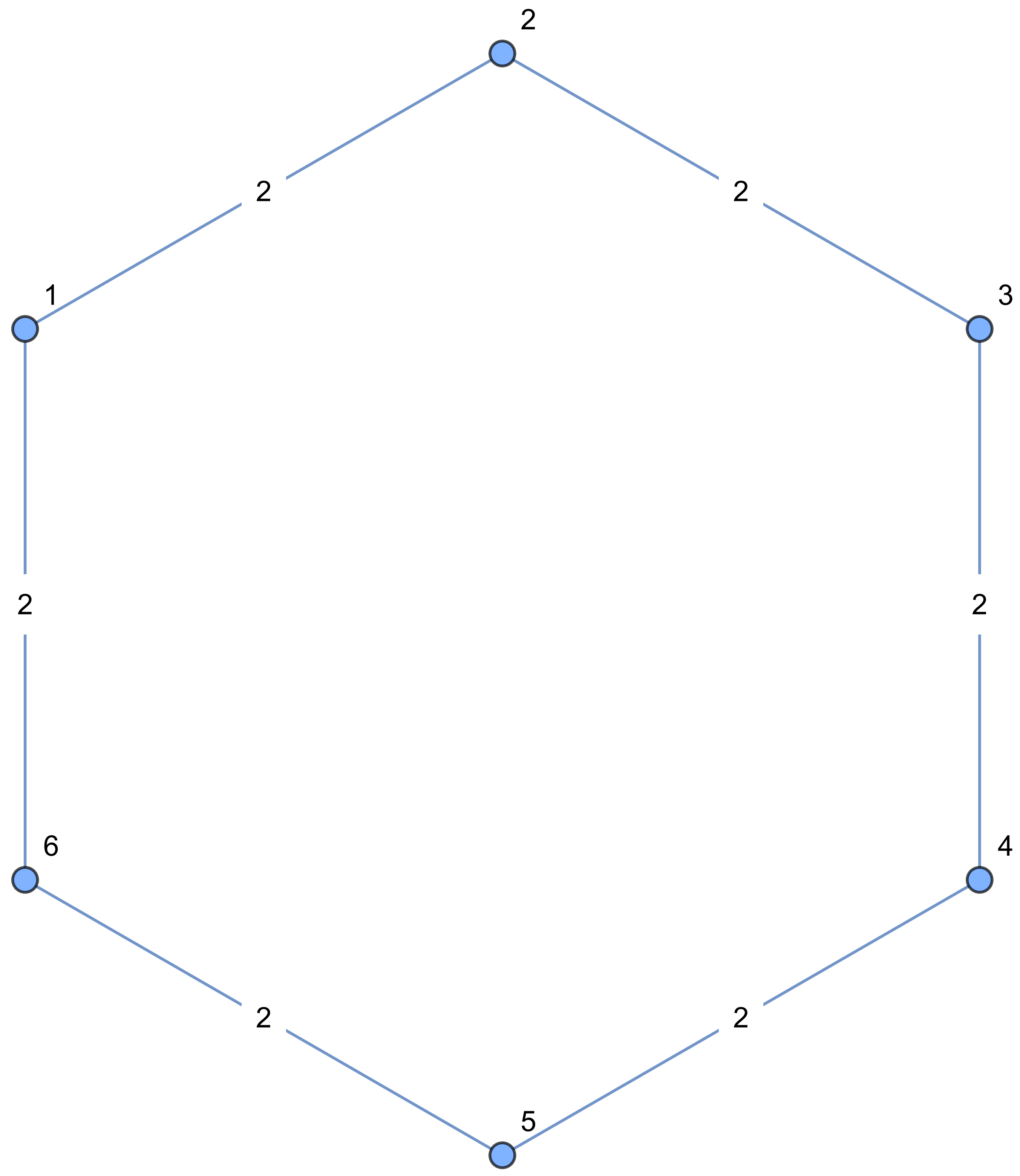
… SIMULTANEOUSLY BY THIS. WHY?

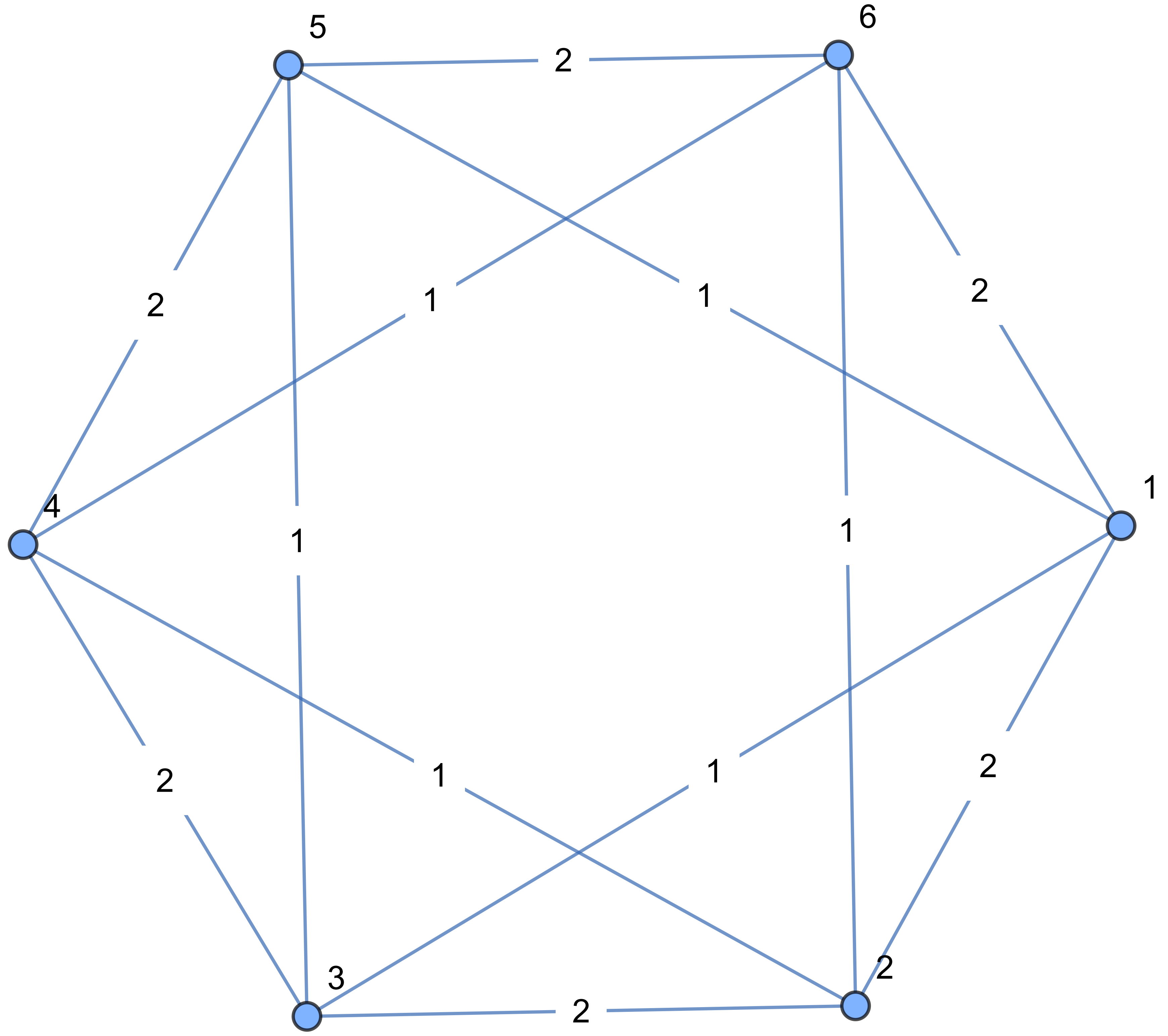
$$S^{-1}MS$$

$$\begin{pmatrix}
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\
\frac{1}{\sqrt{6}} & \frac{e^{\frac{i\pi}{3}}}{\sqrt{6}} & \frac{e^{\frac{2i\pi}{3}}}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{e^{-\frac{2i\pi}{3}}}{\sqrt{6}} & \frac{e^{-\frac{i\pi}{3}}}{\sqrt{6}} \\
\frac{1}{\sqrt{6}} & \frac{e^{\frac{2i\pi}{3}}}{\sqrt{6}} & \frac{e^{-\frac{2i\pi}{3}}}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{e^{\frac{2i\pi}{3}}}{\sqrt{6}} & \frac{e^{-\frac{2i\pi}{3}}}{\sqrt{6}} \\
\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\
\frac{1}{\sqrt{6}} & \frac{e^{-\frac{2i\pi}{3}}}{\sqrt{6}} & \frac{e^{\frac{2i\pi}{3}}}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{e^{-\frac{2i\pi}{3}}}{\sqrt{6}} & \frac{e^{\frac{2i\pi}{3}}}{\sqrt{6}} \\
\frac{1}{\sqrt{6}} & \frac{e^{-\frac{i\pi}{3}}}{\sqrt{6}} & \frac{e^{-\frac{2i\pi}{3}}}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{e^{\frac{2i\pi}{3}}}{\sqrt{6}} & \frac{e^{\frac{i\pi}{3}}}{\sqrt{6}}
\end{pmatrix}$$

Translational Symmetry

(Abelian Symmetry)





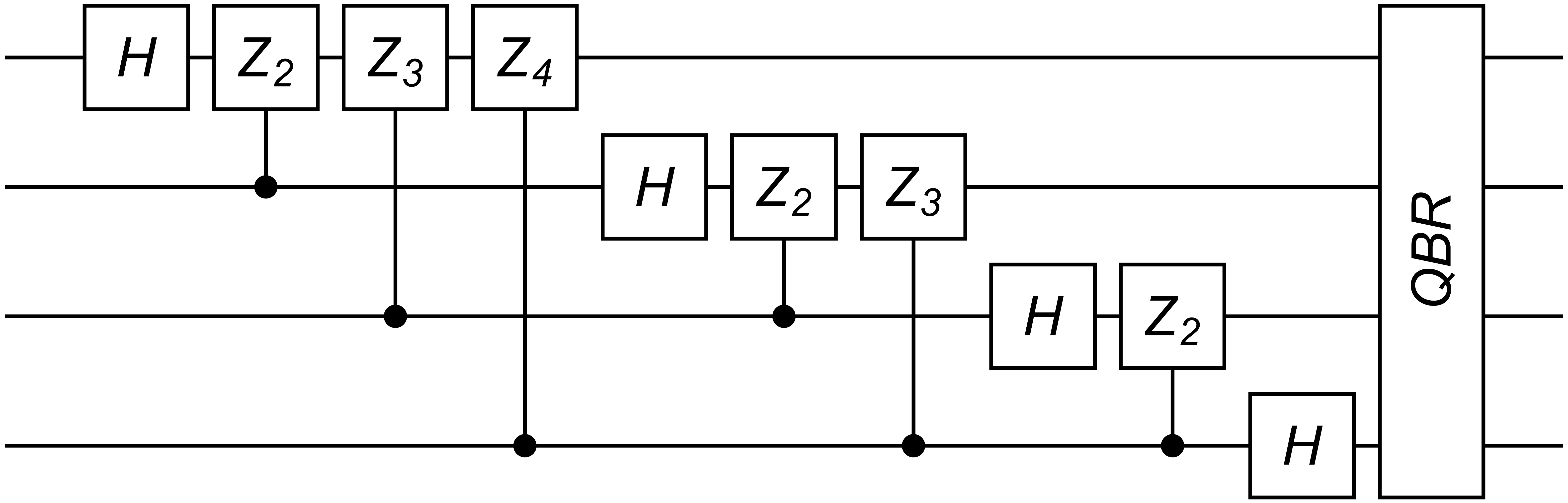
Quantum Fourier Transform

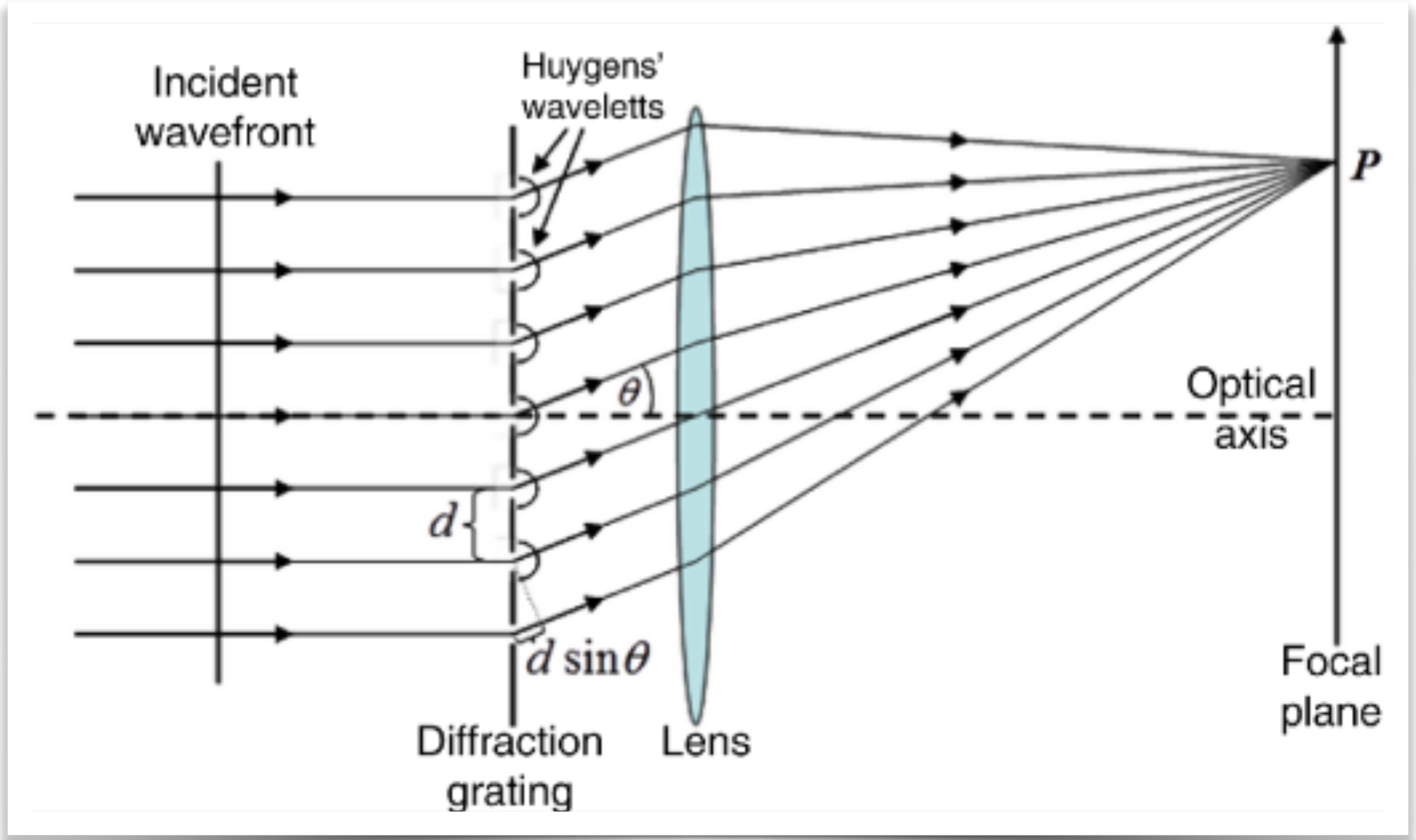
(Abelian Symmetry)

$$|x_j\rangle \mapsto |p_j\rangle := \frac{1}{\sqrt{N}} \sum_k |x_k\rangle \exp\left(i\omega_k p_j\right)$$

$$\omega_k := \frac{2\pi k}{N}$$

QFT CIRCUIT


H
Z2
Z3
Z4
H
Z2
Z3
H
Z2
H
QBR


Incident wavefront
Huygens' waveletts
P
θ
Optical axis
d
d sinθ
Diffraction grating
Lens
Focal plane

Why does it matter?

Abelian stabilizer problems (Kitaev, 1996)

EXAMPLE: SIMON'S PROBLEM

Simon (SIAM J Comp., 1997)

- **Given**: $f : \{0,1\}^n \to \{0,1\}^n$ and secrete string $s \in \{0,1\}^n$
- **Condition**: $f(x) = f(y)$ if and only if $x = y \oplus s$.
- **Goal**: to find secrete s with as few queries to f as possible.

- **Complexity**: queries with $\mathcal{O}\left(2^{n-1}\right)$ inputs ('birthday paradox').
- **Quantum efficiency**: $\mathcal{O}(n)$ assuming an structured 'oracle'.

EXAMPLE: SIMON'S PROBLEM

AS AN ABELIAN HIDDEN SUBGROUP PROBLEM

- Group: $G := Z_2^n$ with $\oplus$ as multiplication
- Subgroup: $H := \{0^n, s\}$
- Cosets: $xH = \{x, x \oplus s : x \in G\}$
- Condition: $f(x) = f(y)$ if and only if $xH = yH$.

Example Simon's Problem

$$\sum_{x \in G} |x\rangle \otimes |0^n\rangle \mapsto \sum_{x \in G} |x\rangle \otimes |f(x)\rangle$$

$$= \sum_{yH} \sum_{h \in H} |y + h\rangle \otimes |f(yH)\rangle$$

$$= \sum_{yH} \left(\sum_{h \in H} |y + h\rangle \right) \otimes |f(yH)\rangle$$

$$\to \sum_{h \in H} |y + h\rangle \quad (y \text{ is known})$$

Bit-Wise QFT & $\mathcal{O}(n)$ measurements

$$
\begin{aligned}
\sum_{h\in H} |y+h\rangle \mapsto \sum_{x\in G}\sum_{h\in H} |x\rangle\, (-1)^{x\cdot(y+h)} \\
= \sum_{x\in G} |x\rangle \left[(-1)^{x\cdot y} + (-1)^{x\cdot(y+s)}\right] \\
= \sum_{x\in G} |x\rangle\, (-1)^{x\cdot y}\left[1 + (-1)^{x\cdot s}\right]
\end{aligned}
$$

Solve the binary linear equation $x_i \cdot s = 0$ for s.

Generalized QFT

Non-Abelian Symmetry

$$|g\rangle \mapsto |\lambda, ij\rangle := \sum_{g\in\mathcal{G}} |g\rangle\, \Gamma^{\lambda}_{ij}(g) \sqrt{\frac{d_\lambda}{|\mathcal{G}|}}$$

$$R(g) = \bigoplus_{\lambda} I_{d_\lambda} \otimes \Gamma^{\lambda}(g)$$

$$\hat{R}(g)\,|h\rangle = |hg^{-1}\rangle$$

What about these?

N.B. They do not commute.

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

They are block-diagonalized …

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 & 0 \\ 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

… Simultaneously by this. Why?

$$S^{-1}MS$$

$$\begin{pmatrix}
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 & 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\
\frac{1}{\sqrt{6}} & -\frac{1}{2\sqrt{3}} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2\sqrt{3}} & -\frac{1}{\sqrt{6}} \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 & 0 & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \\
\frac{1}{\sqrt{6}} & -\frac{1}{2\sqrt{3}} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{6}} \\
\frac{1}{\sqrt{6}} & -\frac{1}{2\sqrt{3}} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{6}} \\
\frac{1}{\sqrt{6}} & -\frac{1}{2\sqrt{3}} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2\sqrt{3}} & -\frac{1}{\sqrt{6}}
\end{pmatrix}$$

Non-Abelian Group

E.g. symmetric group of degree 3

$$\mathcal{S}_3 = \langle \{\pi_{12}, \pi_{123}\} \rangle$$

Generalized QFT

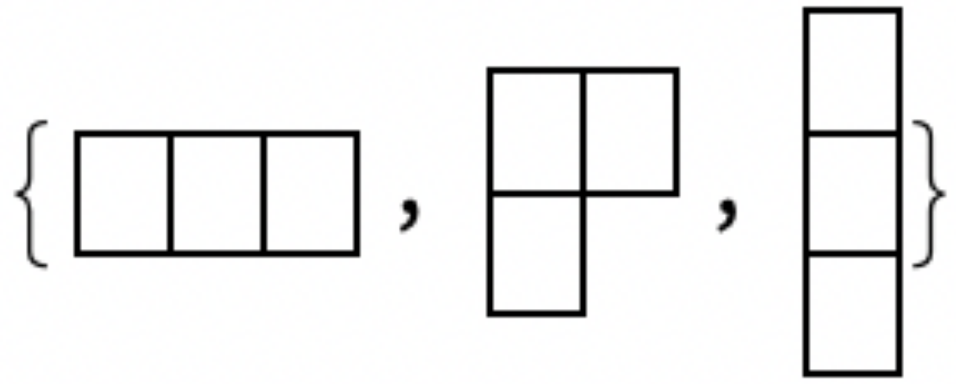
$$|g\rangle \mapsto |\lambda, ij\rangle := \sum_{g \in \mathcal{G}} |g\rangle\, \Gamma^{\lambda}_{ij}(g) \sqrt{\frac{d_\lambda}{|\mathcal{G}|}}$$

$$\hat{R}(g)\,|h\rangle = |hg^{-1}\rangle$$

$$\hat{R}(g)\,|\lambda, ij\rangle = \sum_{k=1}^{d_\lambda} |\lambda, ik\rangle\, \Gamma^{\lambda}_{kj}(g)$$

$$R(g) = \bigoplus_{\lambda} I_{d_\lambda} \otimes \Gamma^{\lambda}(g)$$

The Symmetric Group S_3



$$\left|\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}\,,\ \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}\right\rangle \to \frac{|\pi_0\rangle}{\sqrt{6}}+\frac{|\pi_{12}\rangle}{\sqrt{6}}+\frac{|\pi_{13}\rangle}{\sqrt{6}}+\frac{|\pi_{23}\rangle}{\sqrt{6}}+\frac{|\pi_{123}\rangle}{\sqrt{6}}+\frac{|\pi_{132}\rangle}{\sqrt{6}}$$

$$\left|\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\,,\ \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\right\rangle \to \frac{|\pi_0\rangle}{\sqrt{3}}+\frac{|\pi_{12}\rangle}{\sqrt{3}}-\frac{|\pi_{13}\rangle}{2\sqrt{3}}-\frac{|\pi_{23}\rangle}{2\sqrt{3}}-\frac{|\pi_{123}\rangle}{2\sqrt{3}}-\frac{|\pi_{132}\rangle}{2\sqrt{3}}$$

$$\left|\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\,,\ \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\right\rangle \to -\frac{|\pi_{13}\rangle}{2}+\frac{|\pi_{23}\rangle}{2}+\frac{|\pi_{123}\rangle}{2}-\frac{|\pi_{132}\rangle}{2}$$

$$\left|\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\,,\ \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\right\rangle \to -\frac{|\pi_{13}\rangle}{2}+\frac{|\pi_{23}\rangle}{2}-\frac{|\pi_{123}\rangle}{2}+\frac{|\pi_{132}\rangle}{2}$$

$$\left|\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\,,\ \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\right\rangle \to \frac{|\pi_0\rangle}{\sqrt{3}}-\frac{|\pi_{12}\rangle}{\sqrt{3}}+\frac{|\pi_{13}\rangle}{2\sqrt{3}}+\frac{|\pi_{23}\rangle}{2\sqrt{3}}-\frac{|\pi_{123}\rangle}{2\sqrt{3}}-\frac{|\pi_{132}\rangle}{2\sqrt{3}}$$

$$\left|\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\,,\ \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\right\rangle \to \frac{|\pi_0\rangle}{\sqrt{6}}-\frac{|\pi_{12}\rangle}{\sqrt{6}}-\frac{|\pi_{13}\rangle}{\sqrt{6}}-\frac{|\pi_{23}\rangle}{\sqrt{6}}+\frac{|\pi_{123}\rangle}{\sqrt{6}}+\frac{|\pi_{132}\rangle}{\sqrt{6}}$$

All permutations are block-diagonalized …

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 & 0 \\ 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Why does it matter?

Hidden subgroup problems (still unsolved)

Complexity of QFT over S_n

- Classical FFT, $\mathcal{O}(n^3 n!)$
- Quantum algorithm (More et al., 2004): $\mathcal{O}(n^4 \log n)$
- Quantum algorithm (Kawano et al., 2016): $\mathcal{O}(n^3 \log n)$
- Quantum algorithm (*in preparation*): $\mathcal{O}(n^2 \log n)$?

QFT / Abelian Groups

Discrete Fourier Transform

$$F(p_k) = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} f(x) e^{i p_k x}$$

$$p_k = \frac{2\pi k}{N} \quad (k = 0, 1, \cdots, N-1)$$

Quantum Fourier Transform

$$F(p_k) = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} f(x) e^{ip_k x}$$

$$|p_k\rangle = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} |x\rangle \, e^{ip_k x}$$

$$p_k = \frac{2\pi k}{N} \quad (k = 0, 1, \cdots, N-1)$$

$$|x\rangle := |(x_1x_2\cdots x_n)_2\rangle := |x_1\rangle \otimes |x_2\rangle \otimes \cdots \otimes |x_n\rangle$$

$$\left|2\pi\frac{k}{2^n}\right\rangle := |(0.k_1k_2\cdots k_n)\rangle := |k_1\rangle \otimes |k_2\rangle \otimes \cdots \otimes |k_n\rangle$$

$$|0.k_1k_2\cdots k_n\rangle \to \sum_{x_1=0}^{1}\sum_{x_2=0}^{1}\cdots\sum_{x_n=0}^{1}|x_1x_2\cdots x_n\rangle\, e^{2\pi i(0.k_1k_2\cdots k_n)(x_1x_2\cdots x_n)}$$

$$\sum_{x_1}|x_1\rangle\, e^{2\pi i(0.k_n)x_1} \otimes \sum_{x_2}|x_1\rangle\, e^{2\pi i(0.k_{n-1}k_n)x_2} \otimes \cdots \sum_{x_n}|x_n\rangle\, e^{2\pi i(0.k_1k_2\cdots k_n)x_n}$$

$$\sum_{x_1} |x_1\rangle\, e^{2\pi i(0.k_n)x_1} \otimes \sum_{x_2} |x_1\rangle\, e^{2\pi i(0.k_{n-1}k_n)x_2} \otimes \cdots \sum_{x_n} |x_n\rangle\, e^{2\pi i(0.k_1 k_2 \cdots k_n)x_n}$$

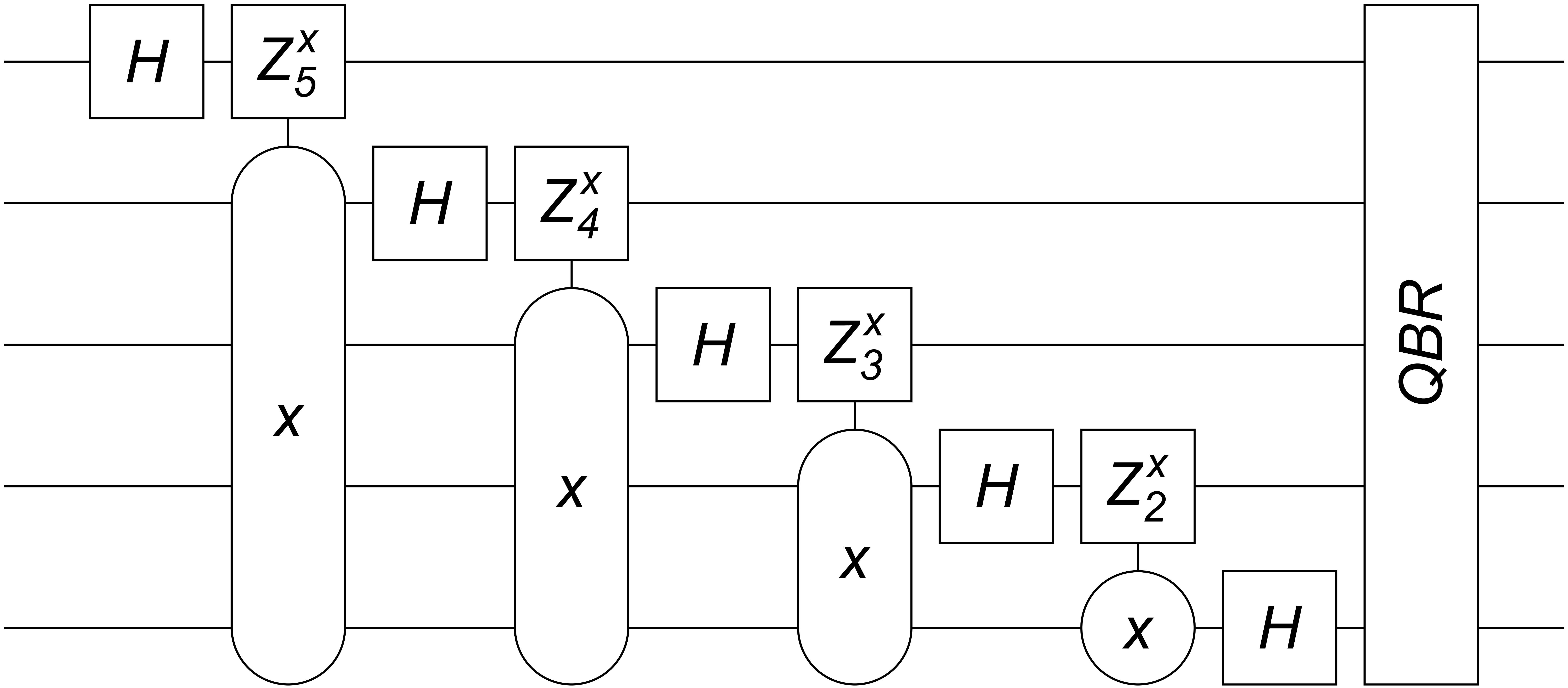
$$\left(|0\rangle + |1\rangle\, e^{2\pi i(0.k_n)}\right) \otimes \left(|0\rangle + |1\rangle\, e^{2\pi i(0.k_{k-1}k_n)}\right) \otimes \cdots \otimes \left(|0\rangle + |1\rangle\, e^{2\pi i(0.k_1 k_2 \cdots k_n)}\right)$$

A product state!

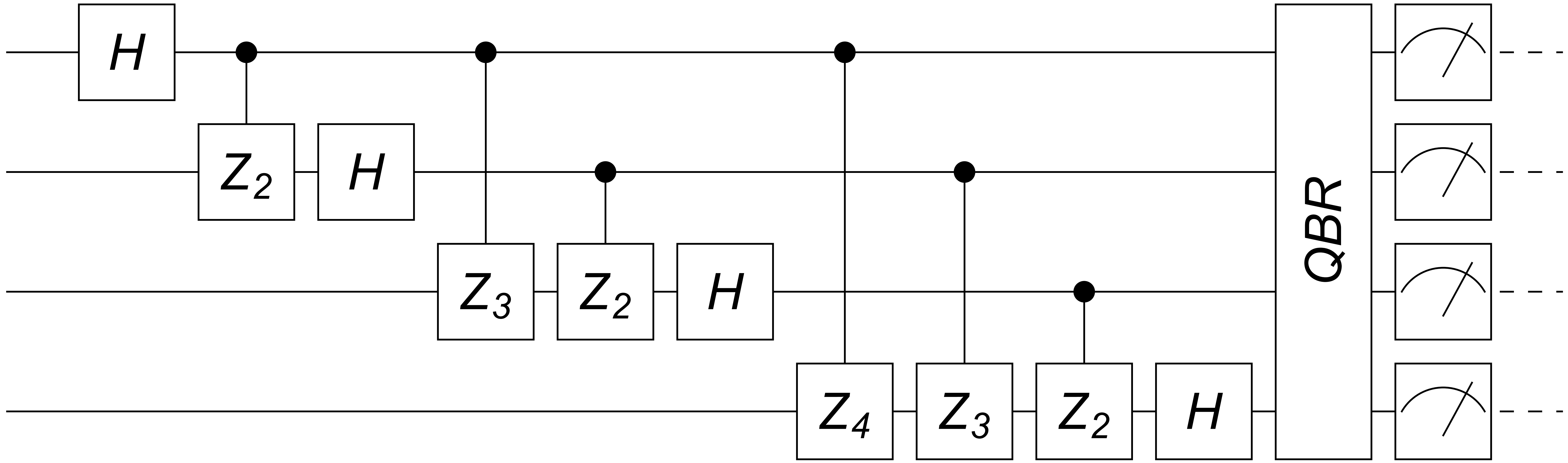
$$\left(|0\rangle + |1\rangle\, e^{2\pi i(0.k_1 k_2 \cdots k_n)}\right) \otimes \cdots \otimes \left(|0\rangle + |1\rangle\, e^{2\pi i(0.k_{k-1}k_n)}\right) \otimes \left(|0\rangle + |1\rangle\, e^{2\pi i(0.k_n)}\right)$$

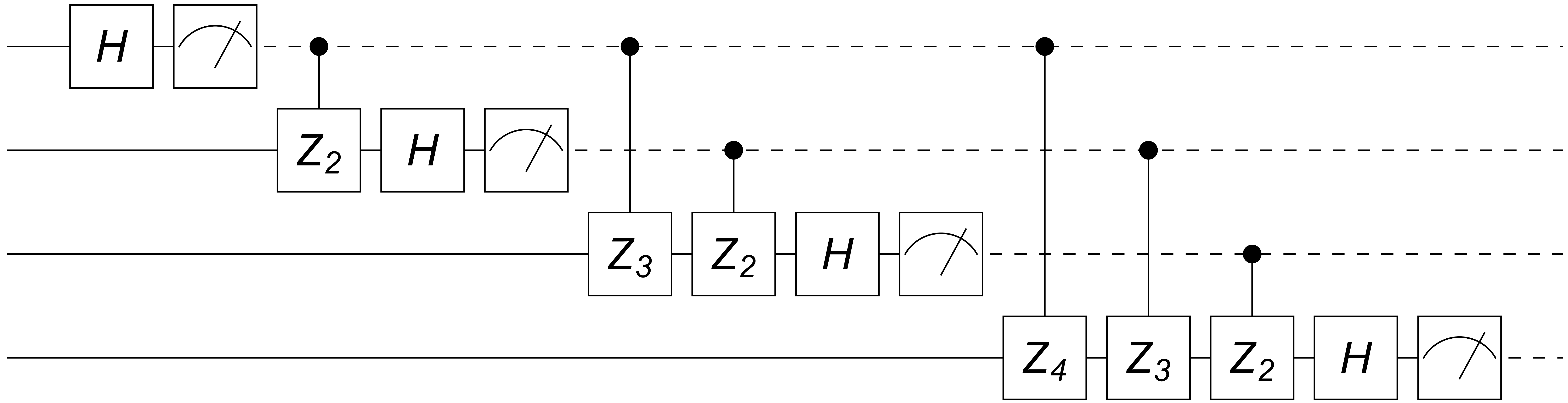
$$|0\rangle + |1\rangle\, e^{2\pi i(0.k_n)} = |0\rangle + |1\rangle\, (-1)^{k_n} = H\, |k_n\rangle$$

$$
\begin{aligned}
|0\rangle + |1\rangle\, e^{2\pi i(0.k_{k-1}k_n)} &= |0\rangle + |1\rangle\, e^{2\pi i(0.k_{k-1})} e^{2\pi i(0.0k_n)} \\
&= R_z\left(2\pi(0.0k_n) = \frac{\pi k_n}{2}\right)\left(|0\rangle + |1\rangle\, e^{2\pi i(0.k_{n-1})}\right) \\
&= R_z\left(\frac{\pi k_n}{2}\right) H\,|k_{n-1}\rangle \\
&= \left[R_z\left(\frac{\pi}{2}\right)\right]^{k_n} H\,|k_{n-1}\rangle
\end{aligned}
$$


H
Z_5^x
x
H
Z_4^x
x
H
Z_3^x
x
H
Z_2^x
x
H
QBR

SEMICLASSICAL QFT

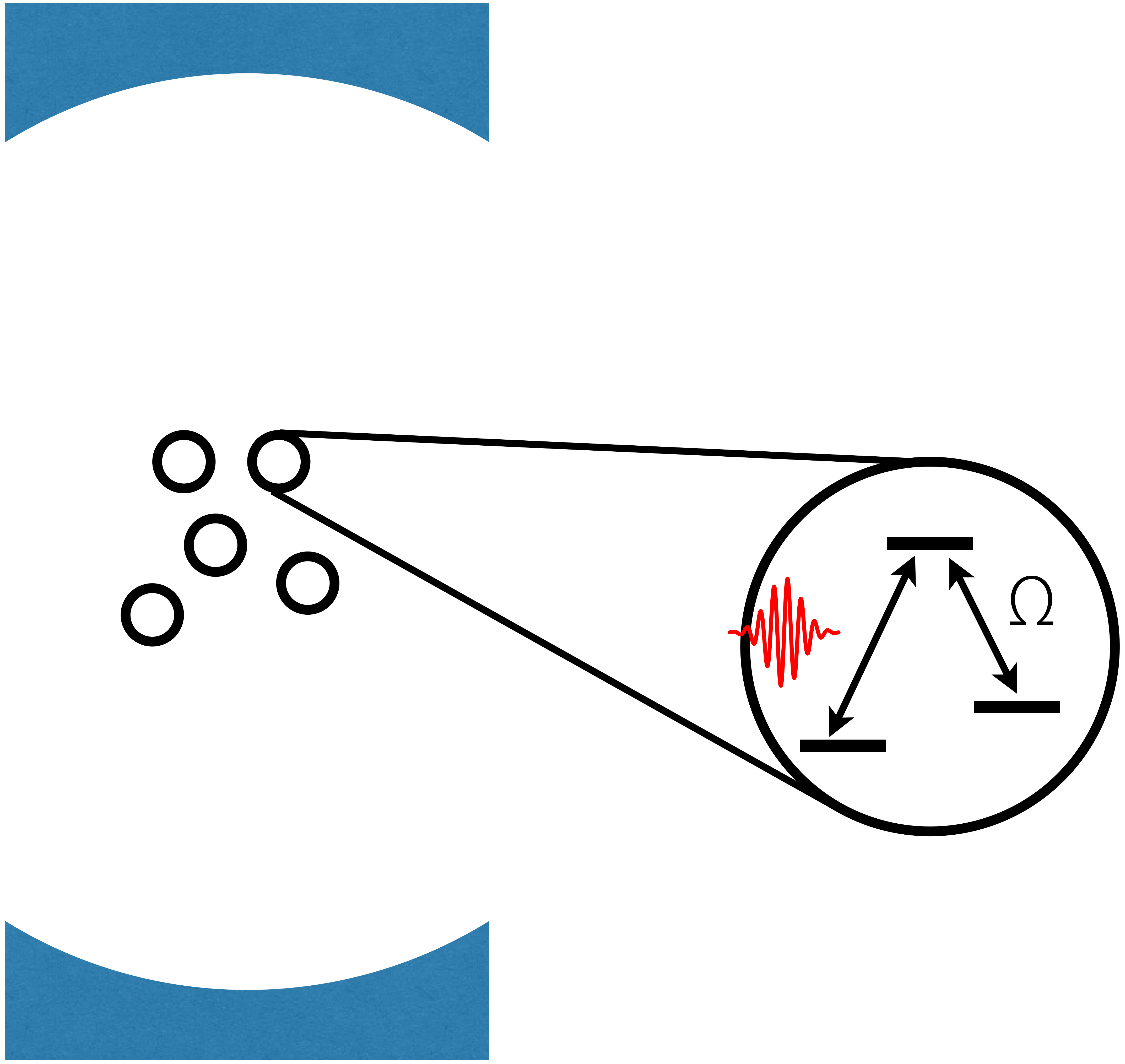


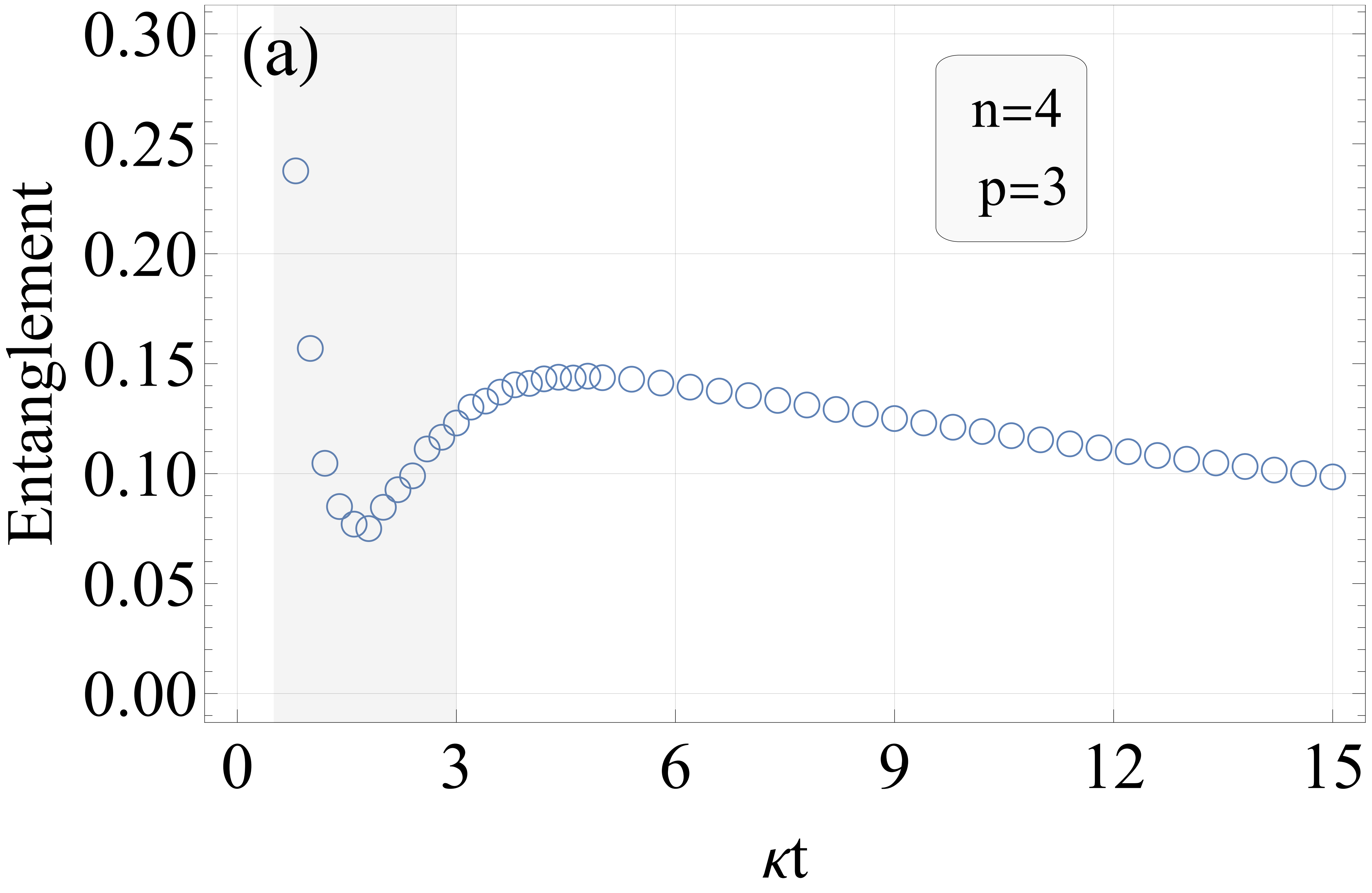


Applications

Quantum Schur Transform

Three-Level Atoms in a Cavity

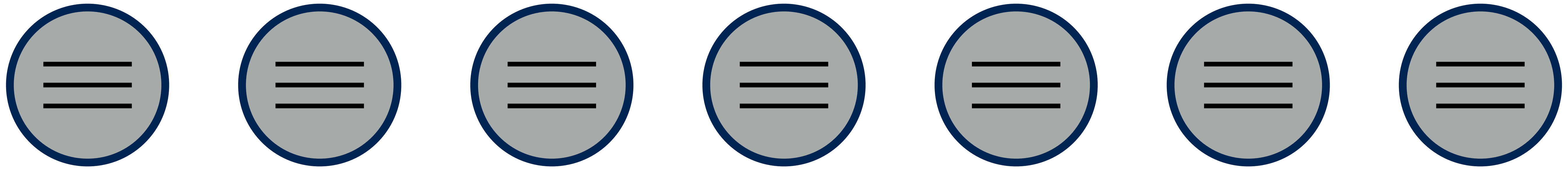




Chen, Chesi, Choi (PRA, 2022)

A System of Many d-Level 'Atoms'

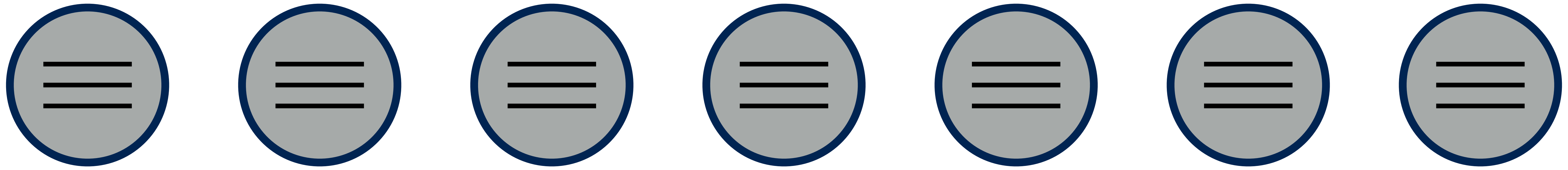
$$\{|s_1\rangle \otimes |s_2\rangle \otimes \cdots \otimes |s_n\rangle : s_k = 0, 1, \cdots, d-1\}$$



$$N := d^n \gg 1$$

A System of Many d-Level 'Atoms'

$$\{|s_1\rangle \otimes |s_2\rangle \otimes \cdots \otimes |s_n\rangle : s_k = 0, 1, \cdots, d-1\}$$



$$N := d^n \gg 1$$

- Can one handle smaller matrices?
- For $d = 2$, SU(2) theory helps.
- Symmetries: permutations (S_n), unitaries (U_d)

Schur Duality

$$\mathcal{V}_d^{\otimes n} \simeq \bigoplus_{\lambda} \mathcal{Y}_\lambda \otimes \mathcal{W}_\lambda$$

$$\Pi \simeq \bigoplus_{\lambda} \Pi^\lambda \otimes I_\lambda$$

$$\Omega \simeq \bigoplus I_\lambda \otimes \Omega^\lambda$$

$\mathcal{Y}_\lambda$: irreducible representation of S_n

$\mathcal{W}_\lambda$: irreducible representation of $U_d^{\otimes n}$

Quantum Schur Transform

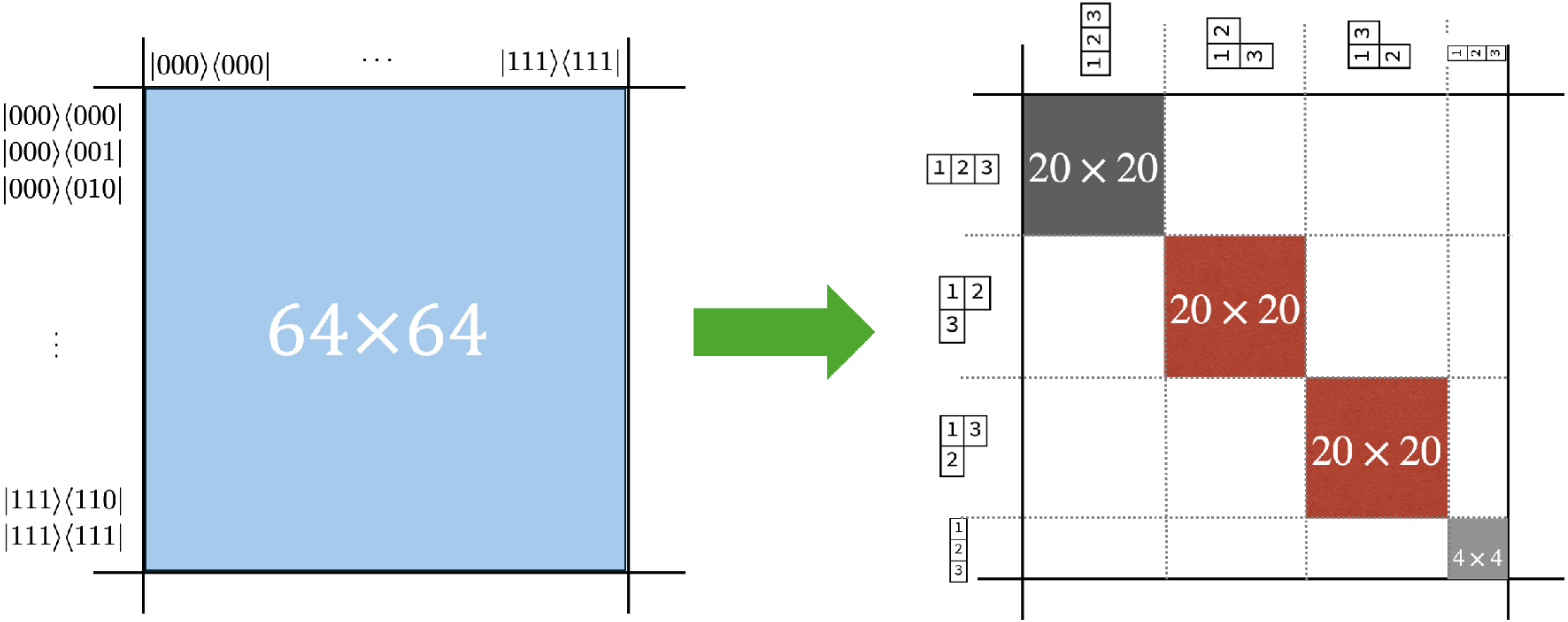
$$|s_1 s_2 \cdots s_n\rangle \mapsto |\lambda; y_\lambda, w_\lambda\rangle$$

$$\hat{\Pi}(\pi)\,|\lambda; y_\lambda, w_\lambda\rangle = \sum_{x_\lambda} |\lambda; x_\lambda, w_\lambda\rangle\, \Pi^\lambda_{xy}(\pi)$$

$$\hat{\Pi}(U)\,|\lambda; y_\lambda, w_\lambda\rangle = \sum_{v_\lambda} |\lambda; y_\lambda, v_\lambda\rangle\, \Omega^\lambda_{vw}(U)$$

Bacon, Chuang, Harrow (2007)
Krovi (2019)

QUANTUM SCHUR TRANSFORM



Bacon, Chuang, Harrow (2007)
Krovi (2019)

Lambda Baryon Λ^0

(A baryon of 3 many 3-flavor quarks)

$$\left|\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline u & s \\ \hline d & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array}\right\rangle = |sud\rangle - |sdu\rangle + |usd\rangle - |dsu\rangle$$

$$\left|\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline u & s \\ \hline d & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array}\right\rangle = 2\,|uds\rangle - 2\,|dus\rangle + |sdu\rangle - |sud\rangle + |usd\rangle - |dsu\rangle$$

Untitled-1.nb

In[1]:=
```
bs = SchurBasis[3, 3];
KeyMap[Map[YoungTableau], bs]
```

Out[2]=

$$\Big\langle\Big|\ \{\boxed{1\,2\,3},\ \boxed{1\,1\,1}\} \to |0,0,0\rangle,\ \{\boxed{1\,2\,3},\ \boxed{1\,1\,2}\} \to \frac{|0,0,1\rangle}{\sqrt{3}} + \frac{|0,1,0\rangle}{\sqrt{3}} + \frac{|1,0,0\rangle}{\sqrt{3}},$$

$$\{\boxed{1\,2\,3},\ \boxed{1\,2\,2}\} \to \frac{|0,1,1\rangle}{\sqrt{3}} + \frac{|1,0,1\rangle}{\sqrt{3}} + \frac{|1,1,0\rangle}{\sqrt{3}},$$

$$\{\boxed{1\,2\,3},\ \boxed{2\,2\,2}\} \to |1,1,1\rangle,\ \{\boxed{1\,2\,3},\ \boxed{1\,1\,3}\} \to \frac{|0,0,2\rangle}{\sqrt{3}} + \frac{|0,2,0\rangle}{\sqrt{3}} + \frac{|2,0,0\rangle}{\sqrt{3}},$$

$$\{\boxed{1\,2\,3},\ \boxed{1\,2\,3}\} \to \frac{|0,1,2\rangle}{\sqrt{6}} + \frac{|0,2,1\rangle}{\sqrt{6}} + \frac{|1,0,2\rangle}{\sqrt{6}} + \frac{|1,2,0\rangle}{\sqrt{6}} + \frac{|2,0,1\rangle}{\sqrt{6}} + \frac{|2,1,0\rangle}{\sqrt{6}},$$

$$\{\boxed{1\,2\,3},\ \boxed{2\,2\,3}\} \to \frac{|1,1,2\rangle}{\sqrt{3}} + \frac{|1,2,1\rangle}{\sqrt{3}} + \frac{|2,1,1\rangle}{\sqrt{3}},$$

$$\{\boxed{1\,2\,3},\ \boxed{1\,3\,3}\} \to \frac{|0,2,2\rangle}{\sqrt{3}} + \frac{|2,0,2\rangle}{\sqrt{3}} + \frac{|2,2,0\rangle}{\sqrt{3}},\ \{\boxed{1\,2\,3},\ \boxed{2\,3\,3}\} \to \frac{|1,2,2\rangle}{\sqrt{3}} + \frac{|2,1,2\rangle}{\sqrt{3}} + \frac{|2,2,1\rangle}{\sqrt{3}},$$

$$\{\boxed{1\,2\,3},\ \boxed{3\,3\,3}\} \to |2,2,2\rangle,\ \left\{\boxed{\begin{smallmatrix}1&2\\3&\end{smallmatrix}},\ \boxed{\begin{smallmatrix}1&1\\2&\end{smallmatrix}}\right\} \to \sqrt{\frac{2}{3}}\,|0,0,1\rangle - \frac{|0,1,0\rangle}{\sqrt{6}} - \frac{|1,0,0\rangle}{\sqrt{6}},$$

$$\left\{\boxed{\begin{smallmatrix}1&2\\3&\end{smallmatrix}},\ \boxed{\begin{smallmatrix}1&2\\2&\end{smallmatrix}}\right\} \to \frac{|0,1,1\rangle}{\sqrt{6}} + \frac{|1,0,1\rangle}{\sqrt{6}} - \sqrt{\frac{2}{3}}\,|1,1,0\rangle,\ \left\{\boxed{\begin{smallmatrix}1&2\\3&\end{smallmatrix}},\ \boxed{\begin{smallmatrix}1&1\\3&\end{smallmatrix}}\right\} \to \sqrt{\frac{2}{3}}\,|0,0,2\rangle - \frac{|0,2,0\rangle}{\sqrt{6}} - \frac{|2,0,0\rangle}{\sqrt{6}},$$

$$\left\{\boxed{\begin{smallmatrix}1&2\\3&\end{smallmatrix}},\ \boxed{\begin{smallmatrix}1&2\\3&\end{smallmatrix}}\right\} \to \frac{|0,1,2\rangle}{\sqrt{3}} - \frac{|0,2,1\rangle}{2\sqrt{3}} + \frac{|1,0,2\rangle}{\sqrt{3}} - \frac{|1,2,0\rangle}{2\sqrt{3}} - \frac{|2,0,1\rangle}{2\sqrt{3}} - \frac{|2,1,0\rangle}{2\sqrt{3}},$$

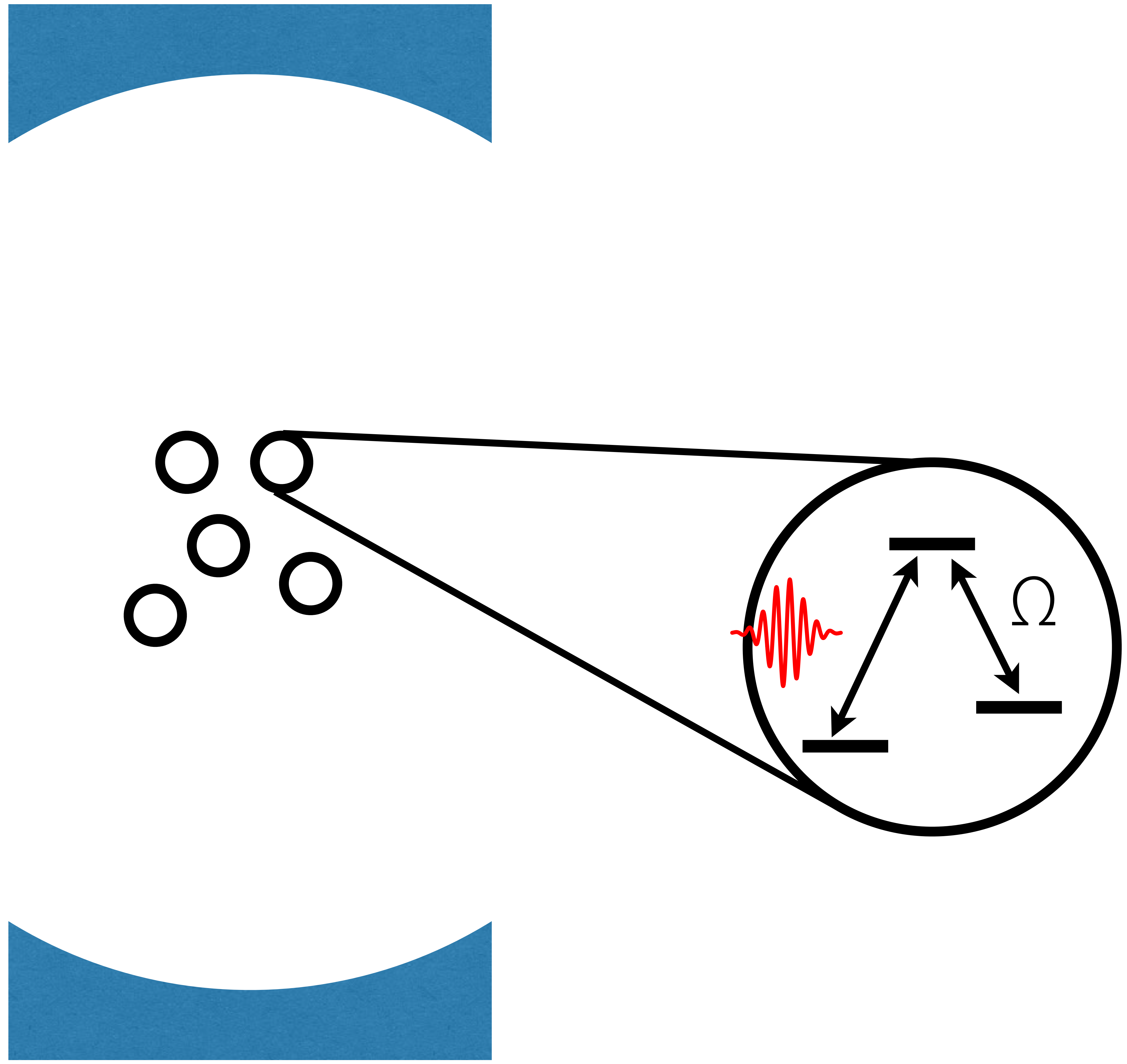
$$\left\{\boxed{\begin{smallmatrix}1&2\\3&\end{smallmatrix}},\ \boxed{\begin{smallmatrix}2&2\\3&\end{smallmatrix}}\right\} \to \sqrt{\frac{2}{3}}\,|1,1,2\rangle - \frac{|1,2,1\rangle}{\sqrt{6}} - \frac{|2,1,1\rangle}{\sqrt{6}},\ \left\{\boxed{\begin{smallmatrix}1&2\\3&\end{smallmatrix}},\ \boxed{\begin{smallmatrix}1&3\\2&\end{smallmatrix}}\right\} \to \frac{|0,2,1\rangle}{2} - \frac{|1,2,0\rangle}{2} + \frac{|2,0,1\rangle}{2} - \frac{|2,1,0\rangle}{2},$$

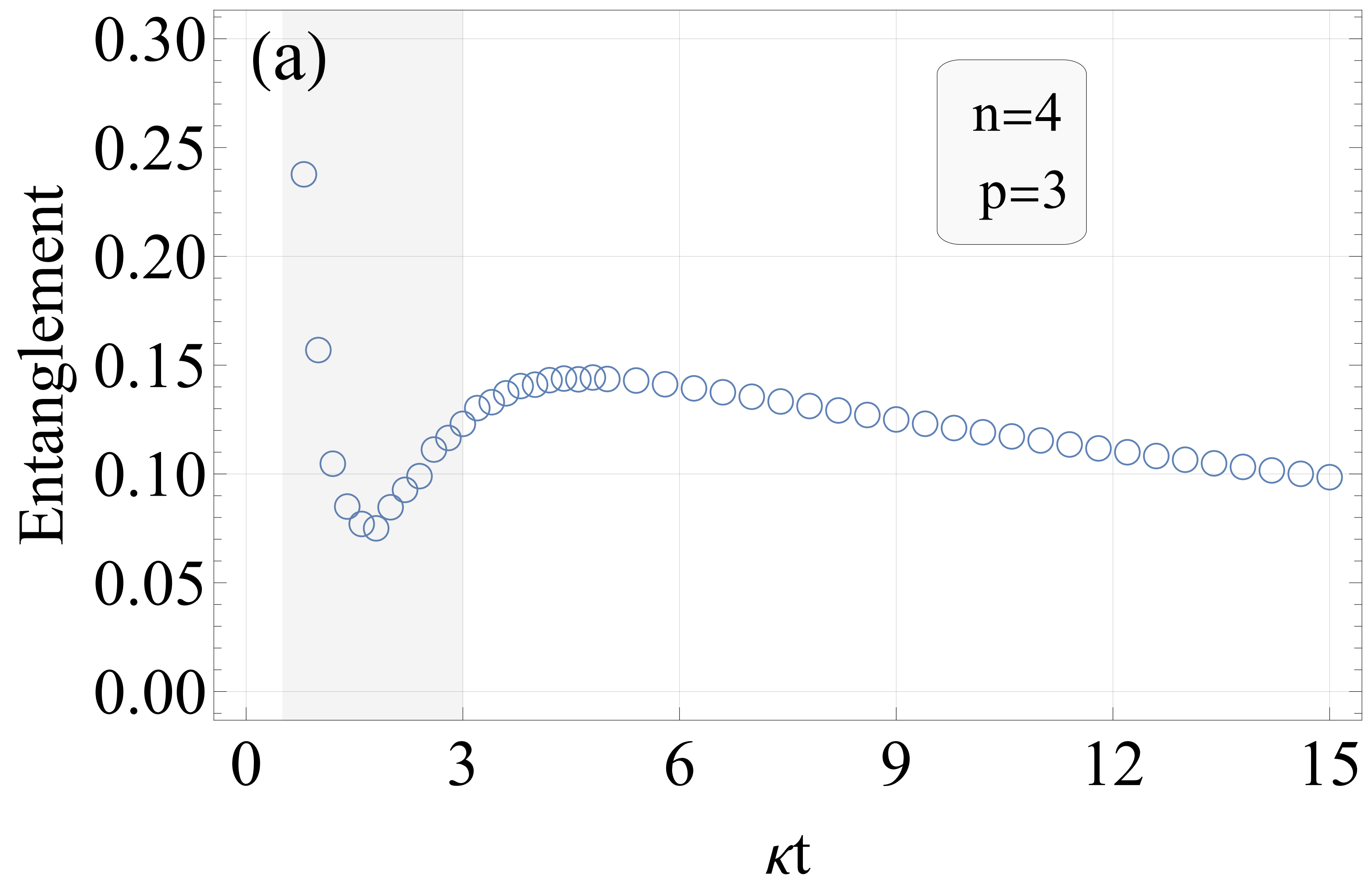
$$\left\{\boxed{\begin{smallmatrix}1&2\\3&\end{smallmatrix}},\ \boxed{\begin{smallmatrix}1&3\\3&\end{smallmatrix}}\right\} \to \frac{|0,2,2\rangle}{\sqrt{6}} + \frac{|2,0,2\rangle}{\sqrt{6}} - \sqrt{\frac{2}{3}}\,|2,2,0\rangle,\ \left\{\boxed{\begin{smallmatrix}1&2\\3&\end{smallmatrix}},\ \boxed{\begin{smallmatrix}2&3\\3&\end{smallmatrix}}\right\} \to \frac{|1,2,2\rangle}{\sqrt{6}} + \frac{|2,1,2\rangle}{\sqrt{6}} - \sqrt{\frac{2}{3}}\,|2,2,1\rangle,$$

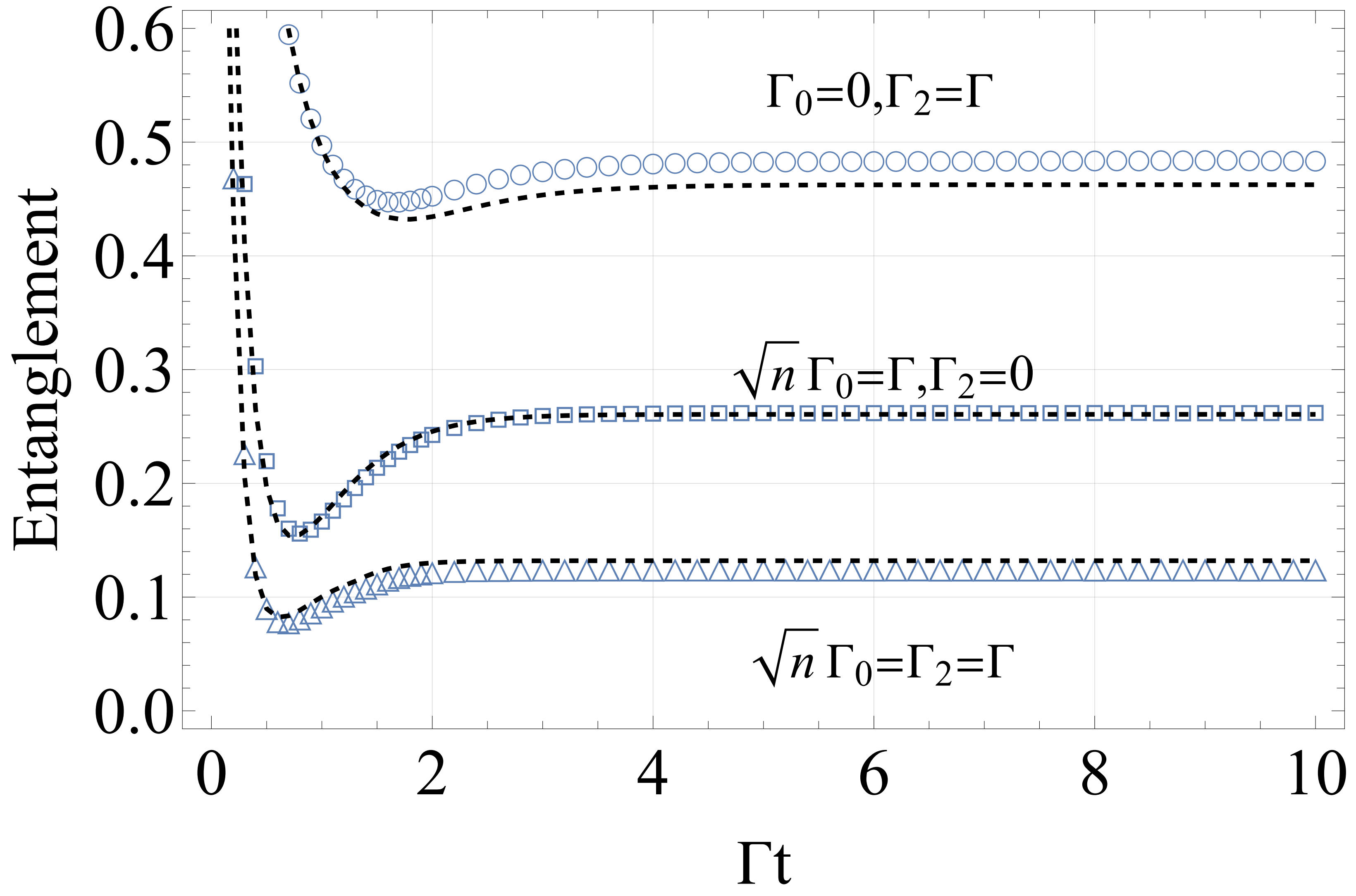
$$\left\{\boxed{\begin{smallmatrix}1&3\\2&\end{smallmatrix}},\ \boxed{\begin{smallmatrix}1&1\\2&\end{smallmatrix}}\right\} \to \frac{|0,1,0\rangle}{\sqrt{2}} - \frac{|1,0,0\rangle}{\sqrt{2}},\ \left\{\boxed{\begin{smallmatrix}1&3\\2&\end{smallmatrix}},\ \boxed{\begin{smallmatrix}1&2\\2&\end{smallmatrix}}\right\} \to \frac{|0,1,1\rangle}{\sqrt{2}} - \frac{|1,0,1\rangle}{\sqrt{2}},\ \left\{\boxed{\begin{smallmatrix}1&3\\2&\end{smallmatrix}},\ \boxed{\begin{smallmatrix}1&1\\3&\end{smallmatrix}}\right\} \to \frac{|0,2,0\rangle}{\sqrt{2}} - \frac{|2,0,0\rangle}{\sqrt{2}},$$

Entanglement Revival in Lambda Matter

Chen, Chesi, Choi (PRA, 2022)



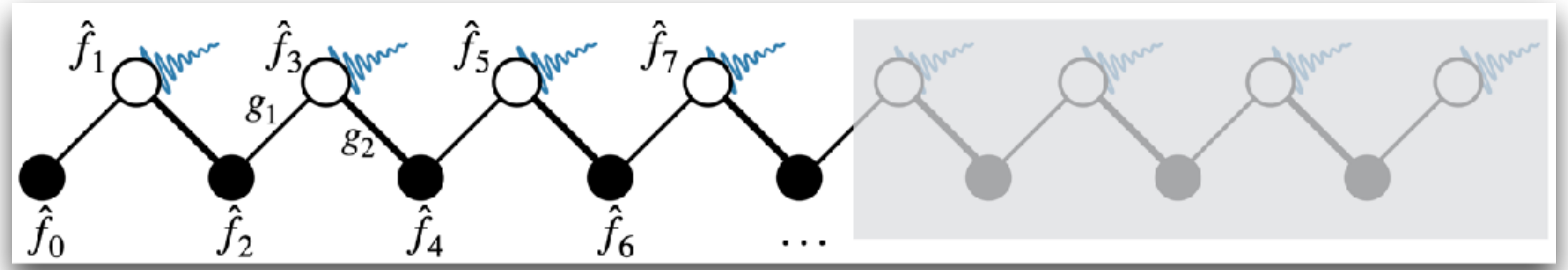


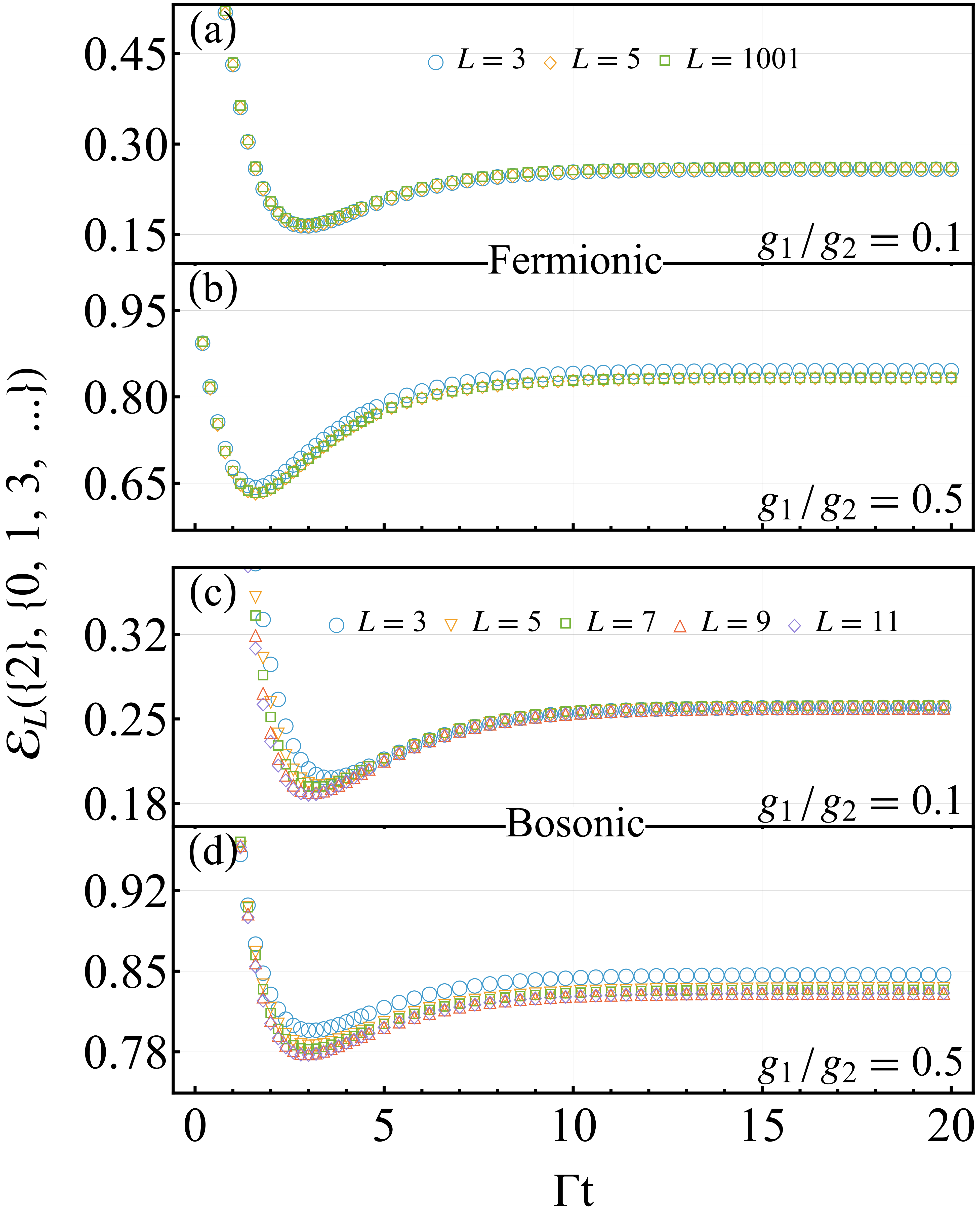


Universal Entanglement Revival

Entanglement-revival dynamics:

- Chen, Chesi, Choi (PRA, 2022)
- Chen, Chesi, Choi (NJP, 2024)
- Chen, Chesi, Choi (PRA, 2025)





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Thank you!