Andreev Bound States

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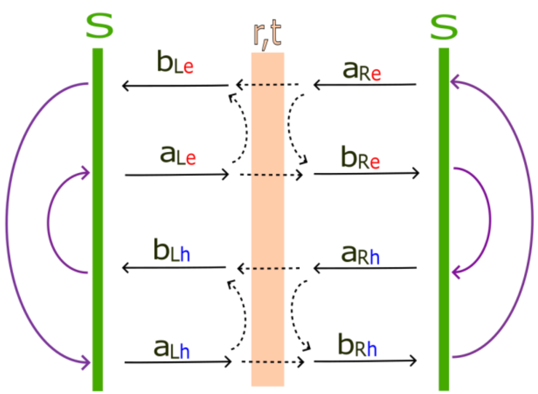
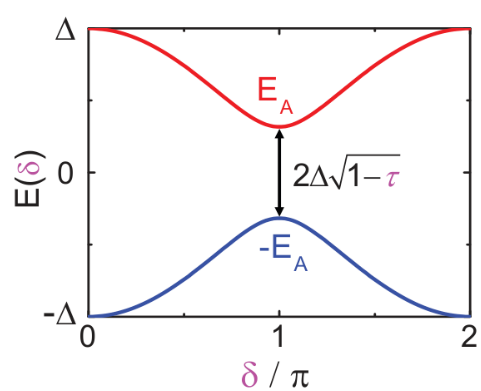
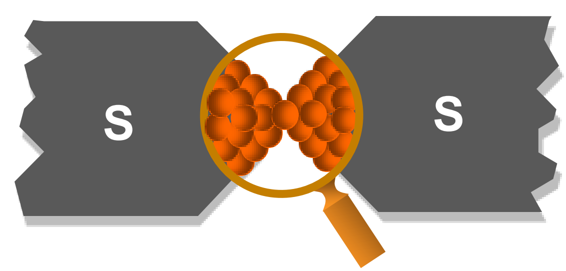
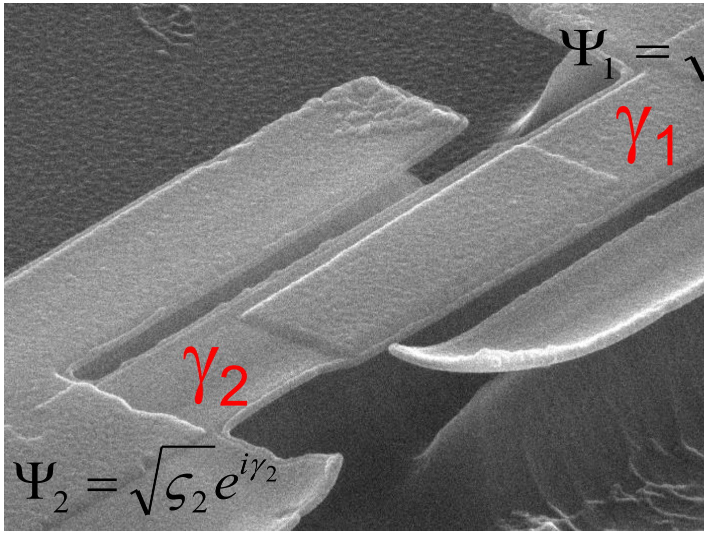






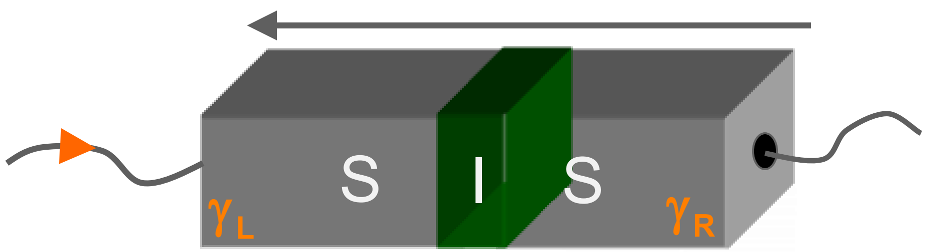
Josephson relations



$$I = I_0 \cdot \sin(\gamma_2 - \gamma_1)$$

$$\frac{d(\gamma_2 - \gamma_1)}{dt} = \frac{2 \cdot e}{\hbar} \cdot V$$

$$V = (\hbar / 2e)\dot{\gamma}$$

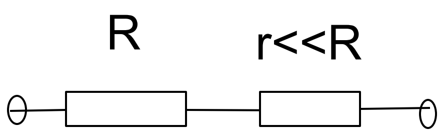


$$I_J = I_0 \sin\gamma$$

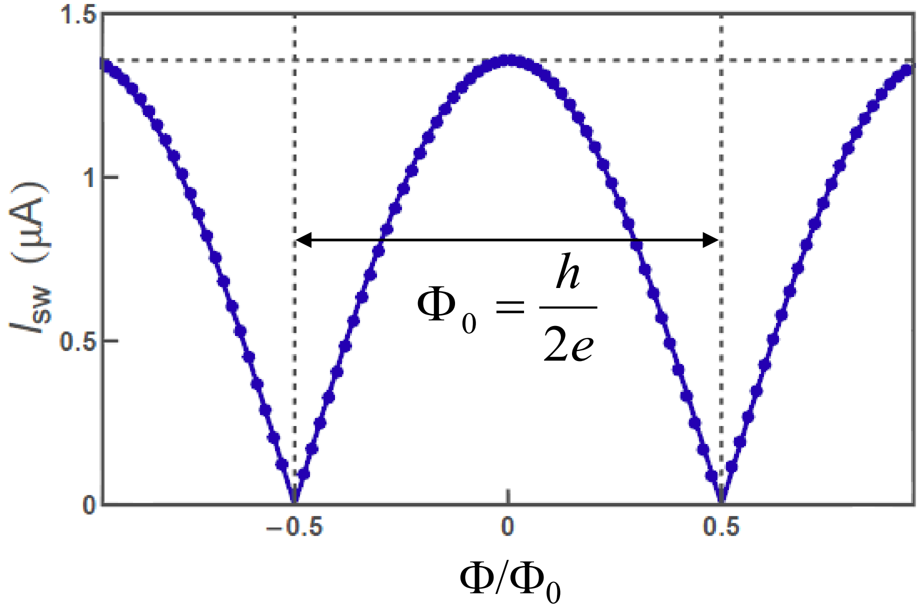
Derivation of Feynman relations:

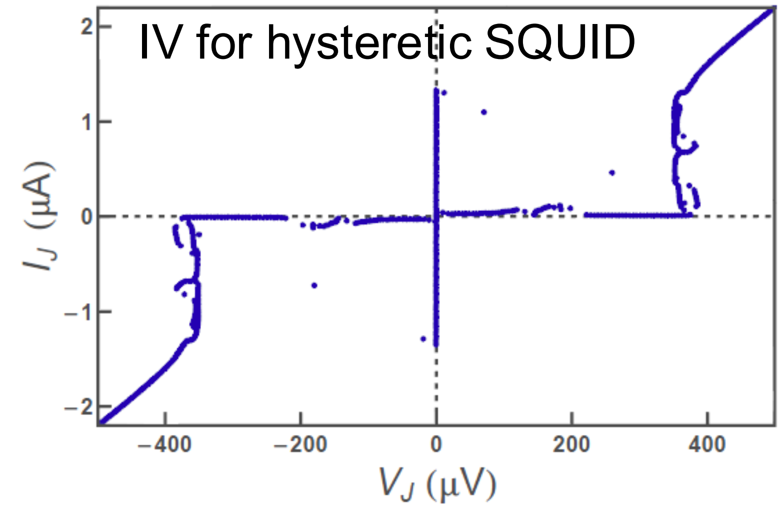
Feynman, vol. III, seminar on superconductivity

Voltage vs. phase

Normal metal	Superconductor
Voltage drop forces current	Phase drop imposes current
 All voltage drops on R	The biggest phase drop in the loop on the weakest weak link

Interference pattern for SQUID



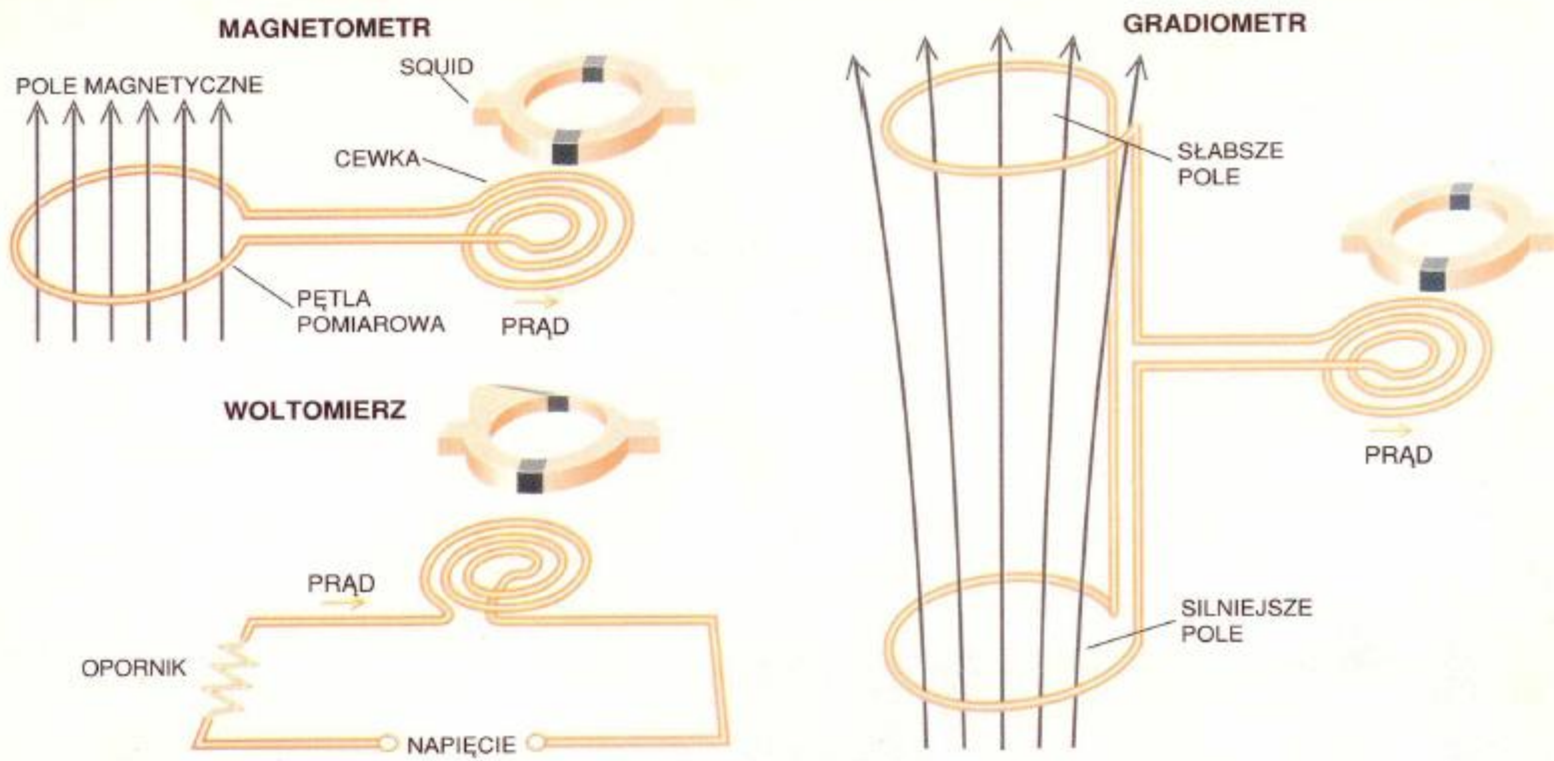


Symmetric Squid is superconducting analog of double slit optical (or electron) interferometer:

applied flux – Φ ⇔ d*sinθ - path difference
Flux quantum – Φ_0 ⇔ λ – wavelength

For symmetric SQUID (2 x JJ): $$I_c = 2 \cdot I_{JJ}^0 \cdot \left|\cos(\pi \cdot \frac{\Phi}{\Phi_0})\right|$$

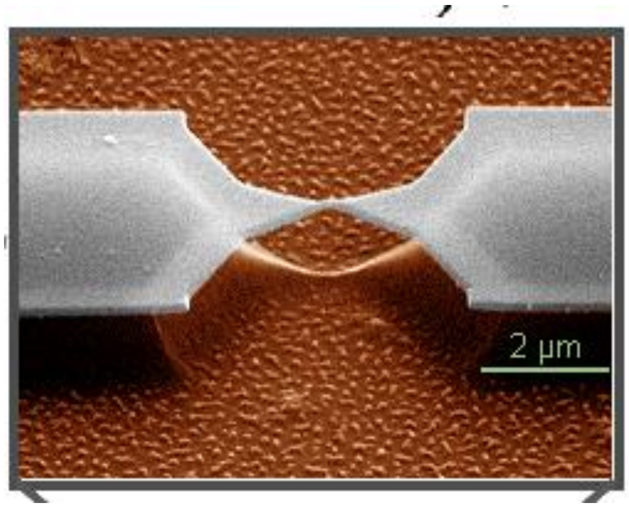
SQUID – various applications

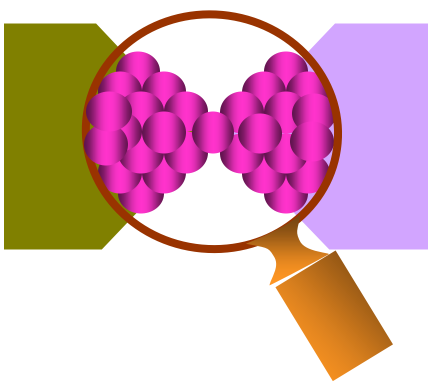


URZĄDZENIA WYKORZYSTUJĄCE SQUID zwykle wymagają dodatkowych elementów. Magnetometr zawiera „transformator strumienia", który składa się z pętli pomiarowej połączonej z cewką sprzężoną z interferometrem. Pole magnetyczne wzbudza prąd w pętli, powodując jego przepływ przez cewkę i powstanie strumienia pola magnetycznego przenikającego interferometr. W gradiometrze dwie oddalone od siebie pętle pomiarowe nawinięte w przeciwnych kierunkach są poddawane jednoczesnemu działaniu pola magnetycznego. Strumień pola magnetycznego przenika interferometr tylko wtedy, gdy natężenie tego pola w obu pętlach jest różne. W woltomierzu mierzone napięcie wywołuje przepływ prądu o natężeniu równym ilorazowi wartości tego napięcia i wielkości oporu połączonego z cewką sprzężoną z nadprzewodnikowym interferometrem kwantowym.

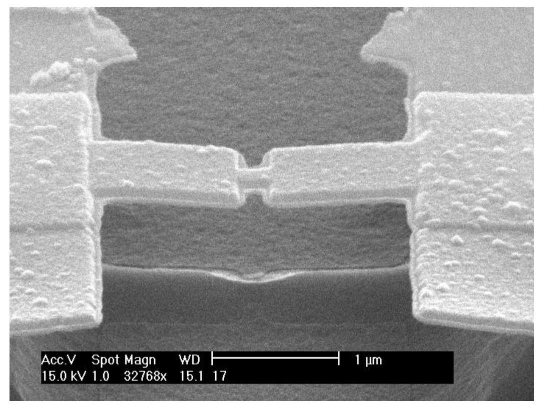
ZOO of Josephson junctions

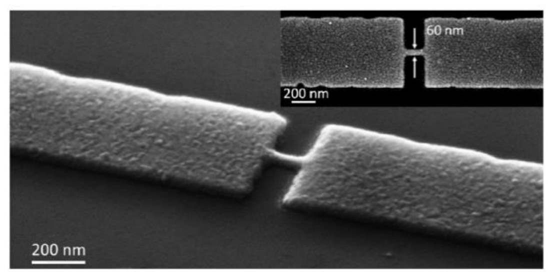
Atomic point contact

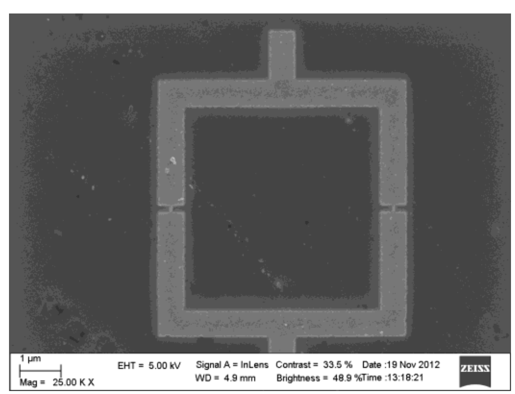




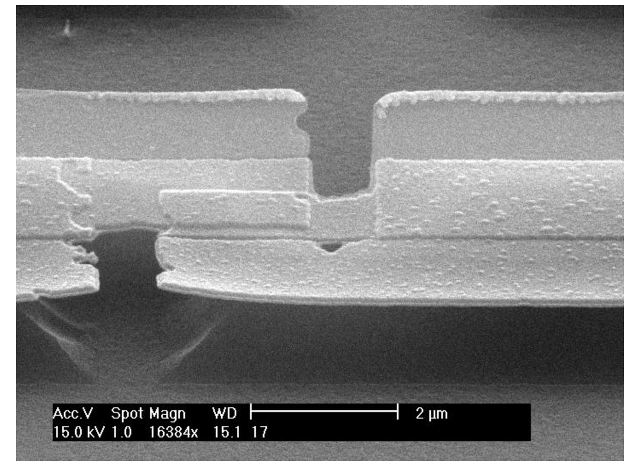
Diffusive weak link







Tunnel junction

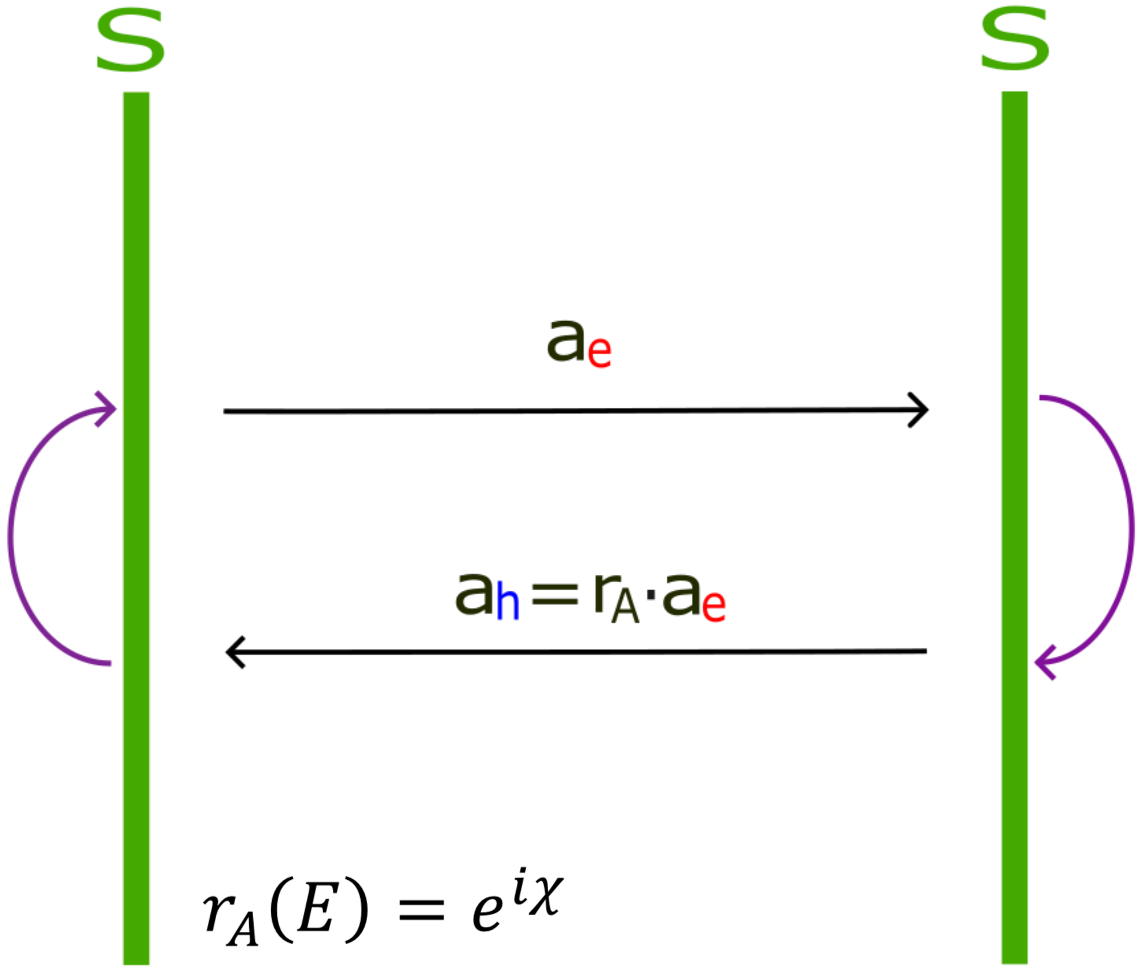


Note: The famous 1st Josephson relation holds only for tunnelling junctions.

But it can be extended to treat all types of JJ !!!

$\Rightarrow$ We will need
ANDREEV BOUND STATES
(our next lecture)

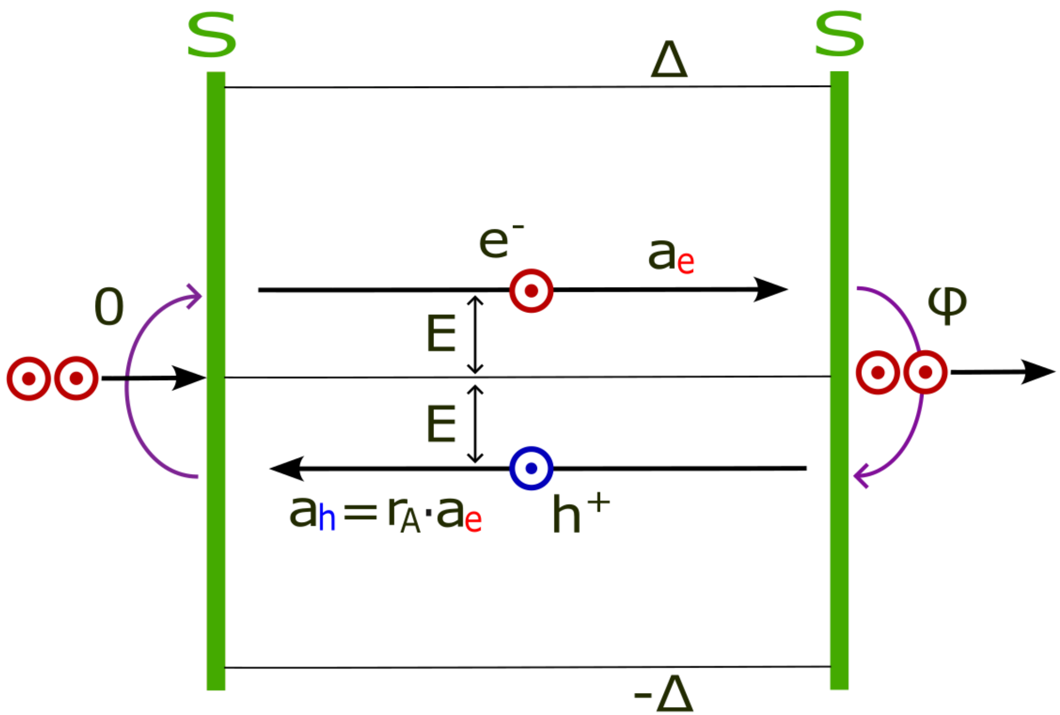
Andreev Reflection



$$r_A(E) = e^{i\chi}$$

$$\chi = -arccos\left(\frac{E}{\Delta}\right) - \varphi$$

Andreev Reflection



Constructive interference happens when:

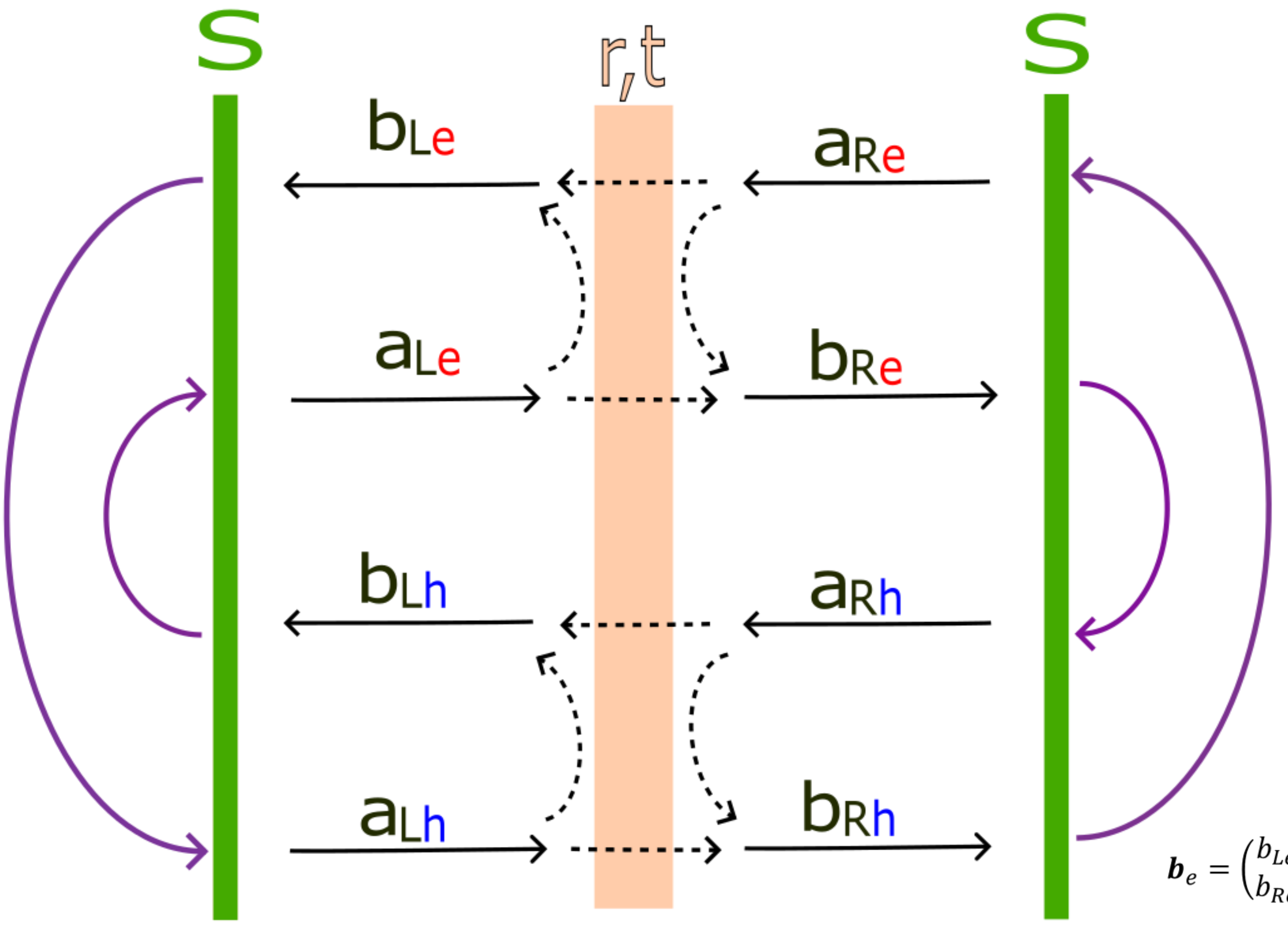
$$\chi_{h\leftarrow e} + \chi_{e\leftarrow h} = 2\pi$$

$$\chi_{h\leftarrow e} = -arccos\left(\frac{E}{\Delta}\right) - \varphi$$

$$\chi_{e\leftarrow h} = -arccos\left(\frac{E}{\Delta}\right) + 0$$

$$r_A(E) = e^{i\chi}$$

Superconducting Junction



$$\boldsymbol{b}_e = \begin{pmatrix} b_{Le} \\ b_{Re} \end{pmatrix} \quad \boldsymbol{b}_h = \begin{pmatrix} b_{Lh} \\ b_{Rh} \end{pmatrix}$$

$$\boldsymbol{a}_e = \begin{pmatrix} a_{Le} \\ a_{Re} \end{pmatrix} \quad \boldsymbol{a}_h = \begin{pmatrix} a_{Lh} \\ a_{Rh} \end{pmatrix}$$

Normal scattering

$$\hat{s}_e = \begin{pmatrix} \sqrt{R}e^{i\theta} & \sqrt{T}e^{i\eta} \\ \sqrt{T}e^{i\eta} & -\sqrt{R}e^{i(2\eta-\theta)} \end{pmatrix}$$

$$\hat{s}_h = \hat{s}_e^*$$

$$\boldsymbol{b}_e = \begin{pmatrix} b_{Le} \\ b_{Re} \end{pmatrix} = \hat{s}_e \begin{pmatrix} a_{Le} \\ a_{Re} \end{pmatrix}$$

$$\begin{pmatrix} \boldsymbol{b}_e \\ \boldsymbol{b}_h \end{pmatrix} = \begin{pmatrix} \hat{s}_e & 0 \\ 0 & \hat{s}_h \end{pmatrix} \begin{pmatrix} \boldsymbol{a}_e \\ \boldsymbol{a}_h \end{pmatrix}$$

Tells how a amplitudes
are converted into b amplitudes

Andreev Reflection

$$\hat{s}_{he} = \begin{pmatrix} e^{i\chi_{heL}} & 0 \\ 0 & e^{i\chi_{heR}} \end{pmatrix}$$

$$\hat{s}_{eh} = \begin{pmatrix} e^{i\chi_{ehL}} & 0 \\ 0 & e^{i\chi_{ehR}} \end{pmatrix}$$

$$\boldsymbol{a}_h = \begin{pmatrix} a_{Lh} \\ a_{Rh} \end{pmatrix} = \hat{s}_{he} \begin{pmatrix} b_{Le} \\ b_{Re} \end{pmatrix}$$

$$\begin{pmatrix} \boldsymbol{a}_e \\ \boldsymbol{a}_h \end{pmatrix} = \begin{pmatrix} 0 & \hat{s}_{eh} \\ \hat{s}_{he} & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{b}_e \\ \boldsymbol{b}_h \end{pmatrix}$$

Tells how b amplitudes
are converted into a amplitudes

Circular relation

$$\begin{pmatrix} \boldsymbol{b}_e \\ \boldsymbol{b}_h \end{pmatrix} = \begin{pmatrix} \hat{s}_e & 0 \\ 0 & \hat{s}_h \end{pmatrix} \begin{pmatrix} \boldsymbol{a}_e \\ \boldsymbol{a}_h \end{pmatrix}$$

$$\begin{pmatrix} \boldsymbol{a}_e \\ \boldsymbol{a}_h \end{pmatrix} = \begin{pmatrix} 0 & \hat{s}_{eh} \\ \hat{s}_{he} & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{b}_e \\ \boldsymbol{b}_h \end{pmatrix}$$

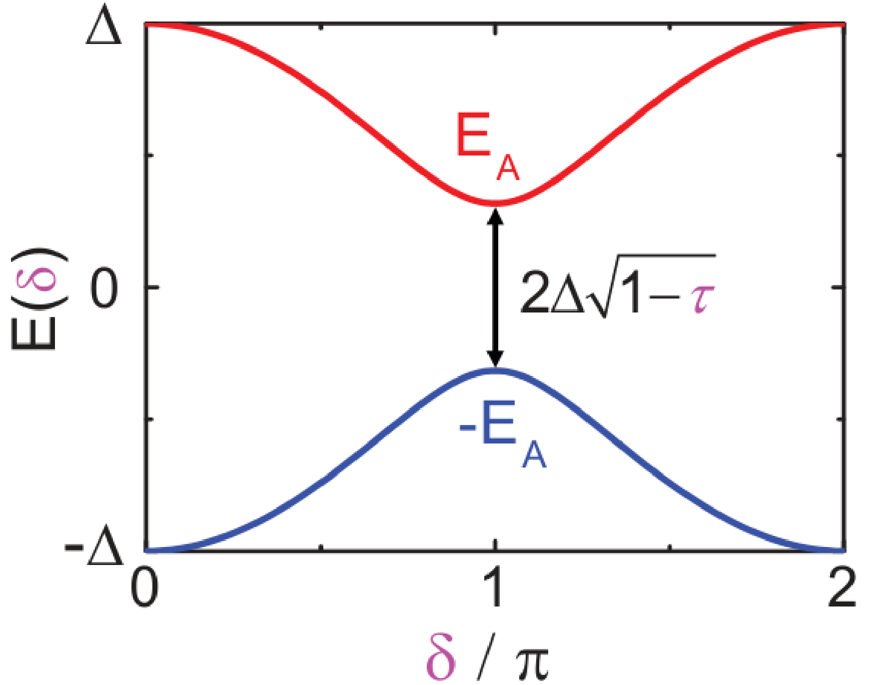
We have to end up in the state where we started

$$\begin{pmatrix} \boldsymbol{b}_e \\ \boldsymbol{b}_h \end{pmatrix} = \begin{pmatrix} \hat{s}_e & 0 \\ 0 & \hat{s}_h \end{pmatrix} \begin{pmatrix} 0 & \hat{s}_{eh} \\ \hat{s}_{he} & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{b}_e \\ \boldsymbol{b}_h \end{pmatrix}$$

Solve for Energy…

Do the algebra in the Wolfram Cloud

Energy of the Andreev Bound State



$$\pm E_A(\delta, \tau) = \pm\Delta\sqrt{1 - \tau \sin^2(\delta/2)}$$

Current-phase relation for a single channel

$$P = VI = \frac{dE}{dt} = \frac{\partial E}{\partial \gamma}\frac{d\gamma}{dt}$$

$$\frac{d\gamma}{dt} = \frac{1}{\varphi_0}V$$

$$I = \frac{\partial E}{\partial \gamma}\frac{1}{\varphi_0}$$

Exercise: Calculate current across JJ for a single channel of transmission tau

The most general current-phase relation (CPR) for a JJ

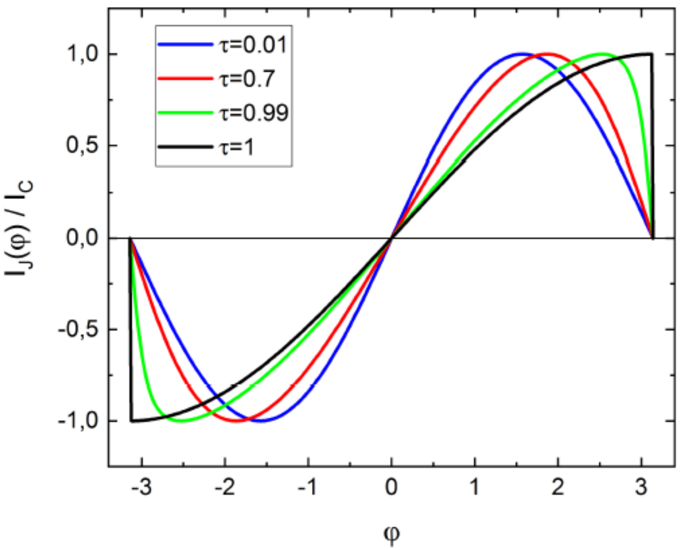
$$I_J(\varphi,T) = \sum_n \frac{e\Delta(T)}{2\hbar} \frac{\tau_n sin\varphi}{\sqrt{1-\tau_n sin^2\frac{\varphi}{2}}} tanh \frac{\Delta(T)\sqrt{1-\tau_n sin^2\frac{\varphi}{2}}}{2k_B T}$$

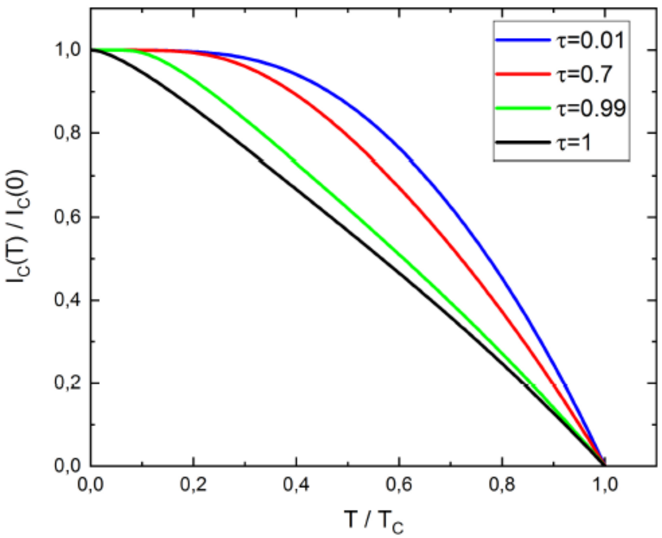
n numbers channels

It reduces to1st Josephson relation if taus->0 for all channels (tunnel limit):

$$I_J(\varphi) = \frac{\pi\Delta(T)}{2eR_N} tanh\frac{\Delta(T)}{2k_B T}\ sin\varphi = i_c\, sin\varphi$$

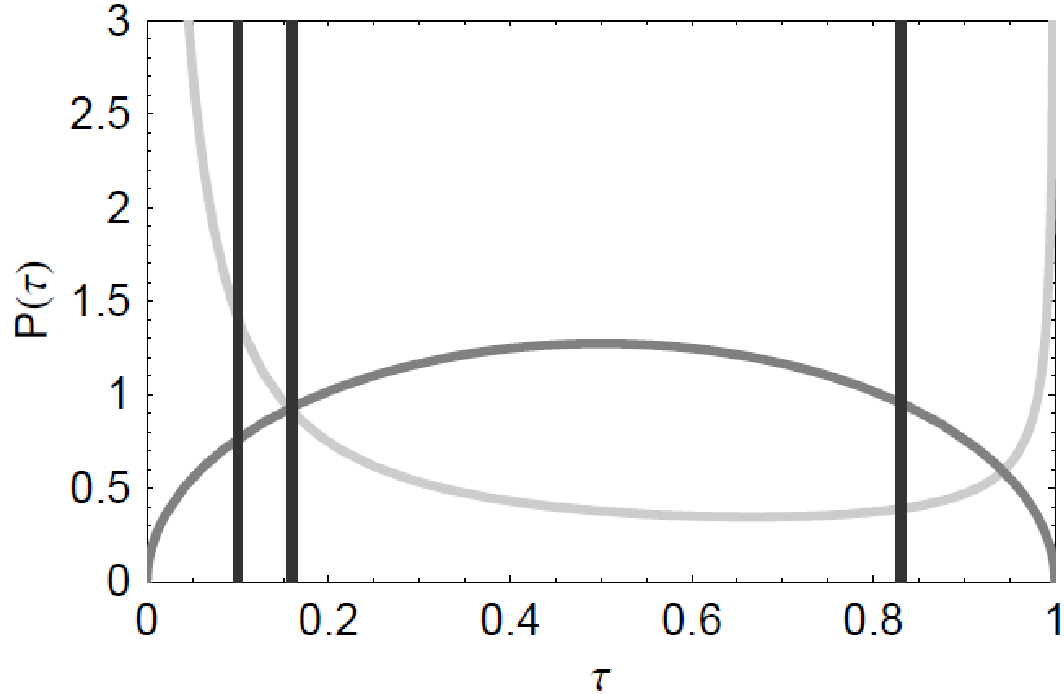
Single-channel CPRs





$$I_J(\varphi) = N_{\text{ch}} \frac{e\Delta(T)}{2\hbar} \frac{\tau \sin\varphi}{\sqrt{1-\tau\sin^2\frac{\varphi}{2}}} \tanh \frac{\Delta(T)\sqrt{1-\tau\sin^2\frac{\varphi}{2}}}{2k_B T}$$

Diffusive weak link:
Dorokhov distribution of channel transmissions

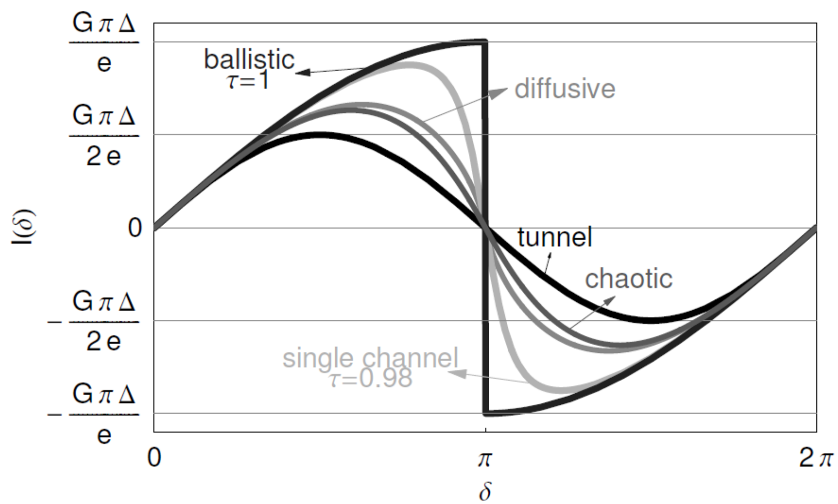


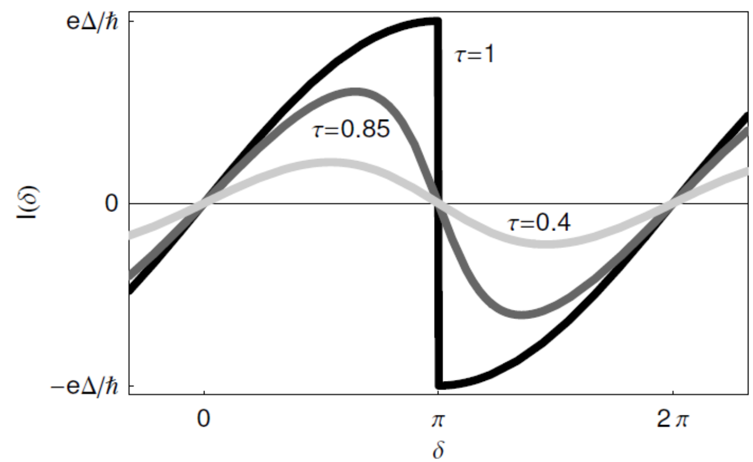
$$I(\delta) = M \int_0^1 I_\tau(\delta) P(\tau) \mathrm{d}\tau = \frac{2Ml_e}{3L} \int_0^1 \frac{\mathrm{d}\tau}{\tau\sqrt{1-\tau}} I_\tau(\delta)$$

Benjamin Huard PhD Thesis (2006), https://iramis.cea.fr/en/spec/gq/theses/

I_cR_N products

$$I_c(0)R_N = \frac{\pi\Delta}{e} \qquad I_c(0)R_N = \frac{\pi\Delta}{2e} \qquad I_c(0)R_N = 2.082\frac{\Delta}{e} \qquad I_c(0)R_N = \frac{1}{1+\sqrt{1-\tau}}\frac{\pi\Delta}{e}$$

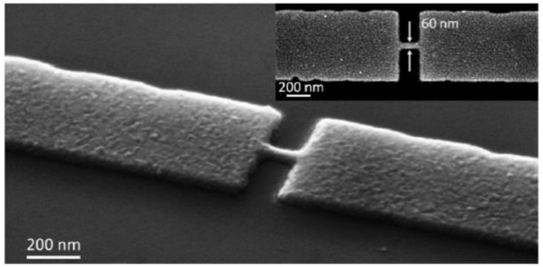


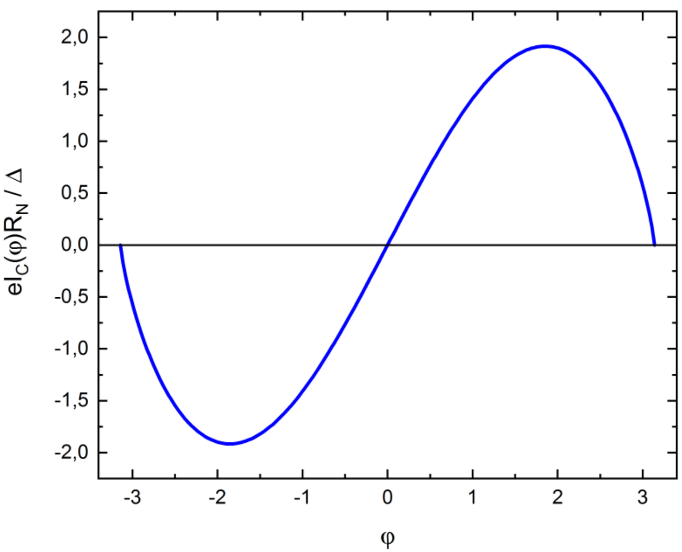


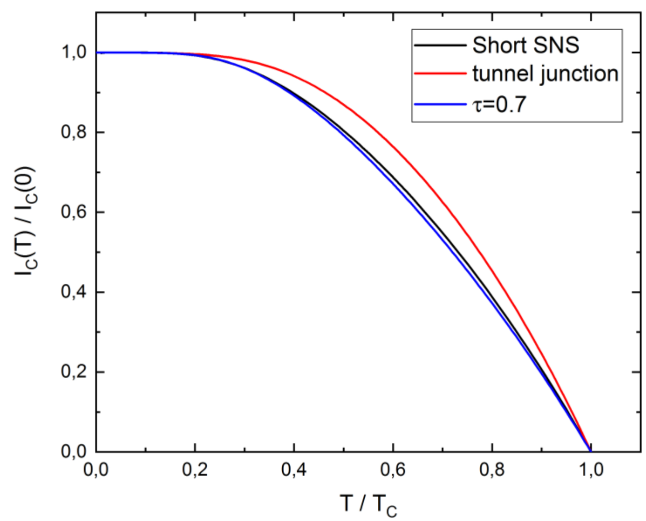
Phase where maximum current is achieved grows with transmission:
It is Pi/2 for tunnel junction with tau->0
and Pi for ballistic channel with tau=1.

Benjamin Huard PhD Thesis (2006), https://iramis.cea.fr/en/spec/gq/theses/

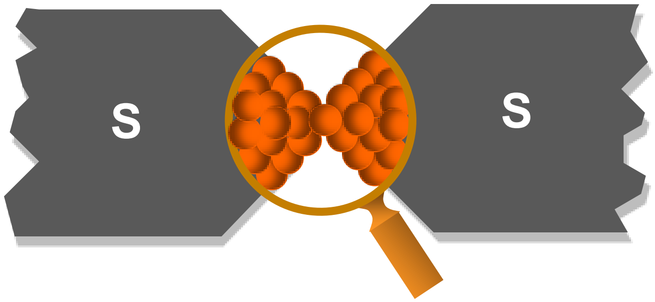
CPR for a diffusive link

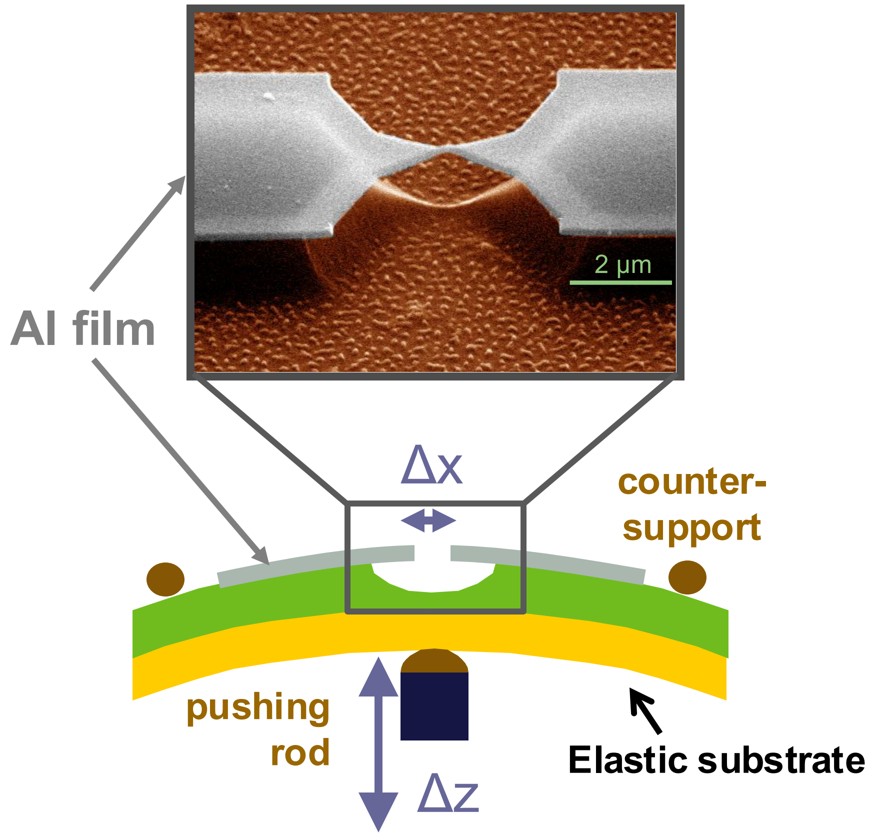


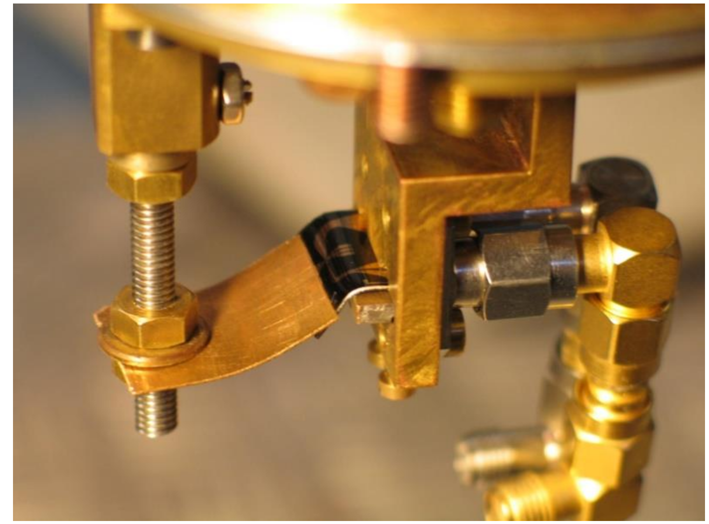




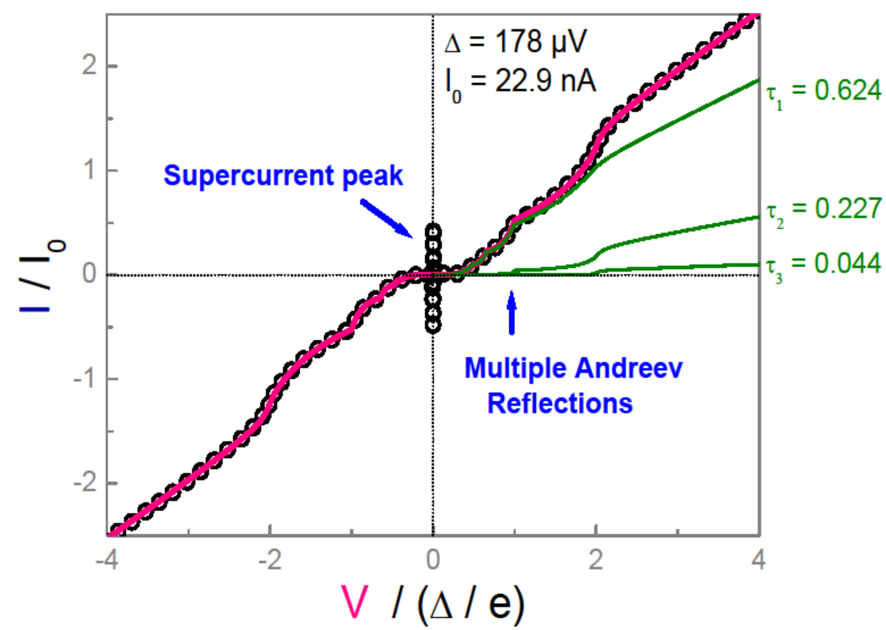
Atomic contact



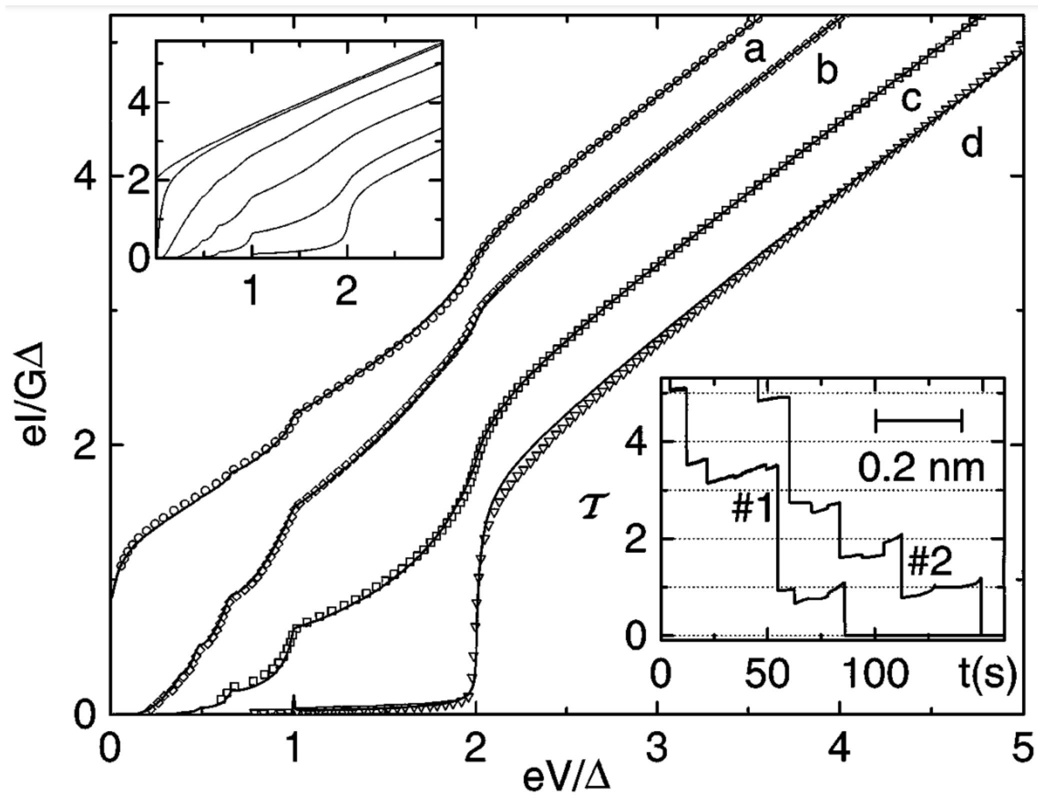




few channels, $\{\tau_i\}$ tunable



Multiple Andreev Reflections (MAR)



The individual channel transmissions and total transmission T obtained from the fits:

(a) $\tau_1 = 0.997$, $\tau_2 = 0.46$, $\tau_3 = 0.29$, T = 1.747;

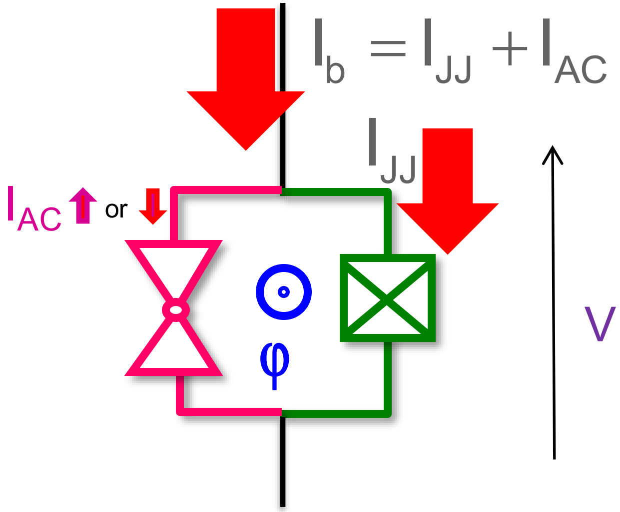
(b) $\tau_1 = 0.74$, $\tau_2 = 0.11$, T = 0.85;

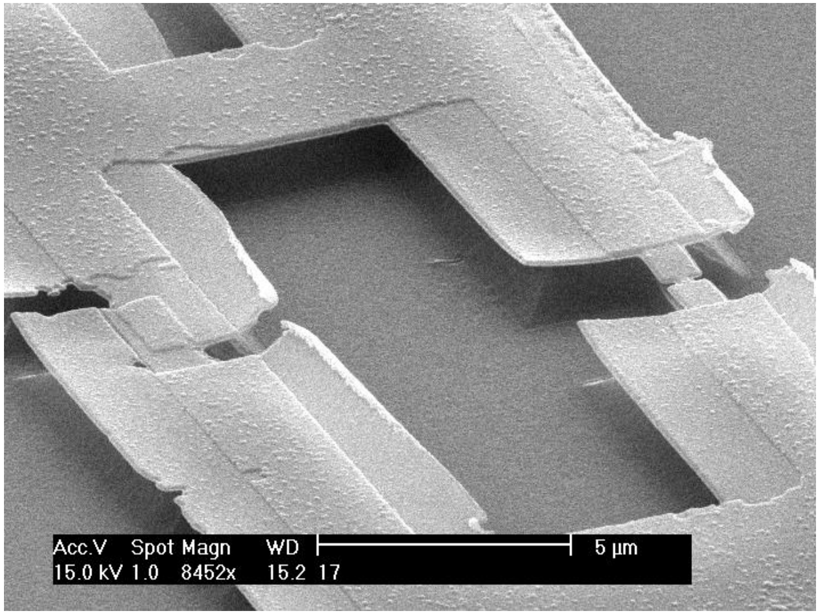
(c) $\tau_1 = 0.46$, $\tau_2 = 0.35$, $\tau_3 = 0.07$, T = 0.88;

(d) T = $\tau_1 = 0.025$.

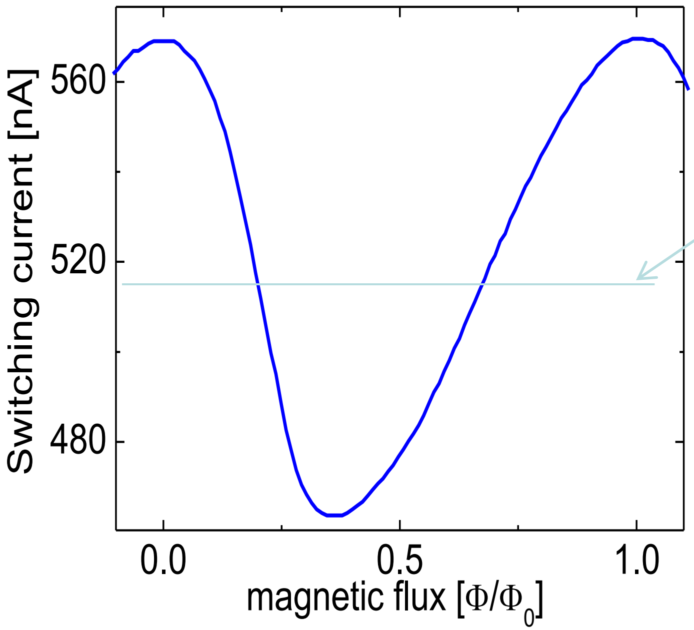
Phys. Rev. Lett. **78**, 3535 (1997)

Atomic Squid…- my PostDoc project





Interference pattern for ATOMIC SQUID



I_0-switching current of junction alone

When SQUID switches, phase across JJ is the same independently of applied magnetic flux => interference pattern is current-phase relation of atomic contact

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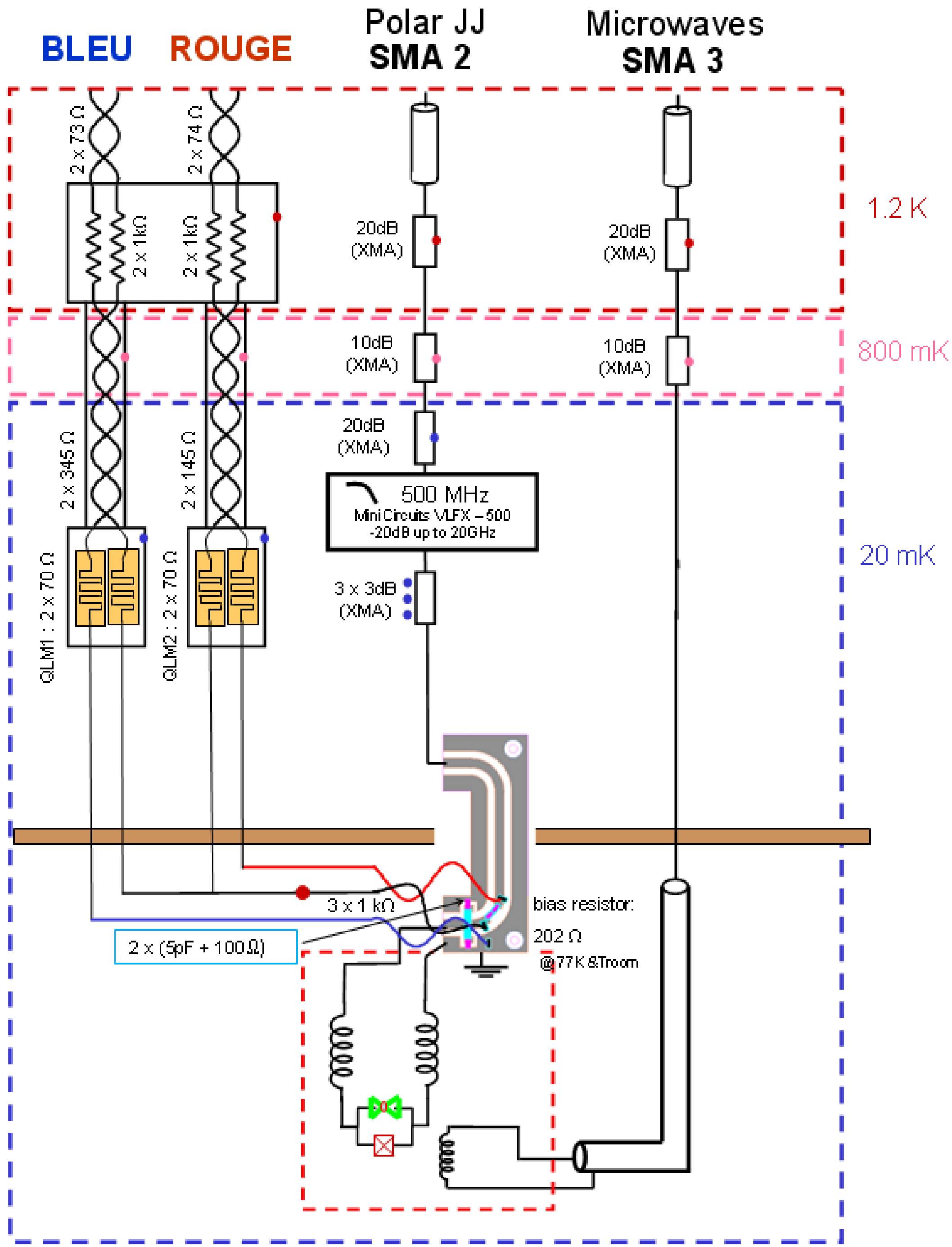
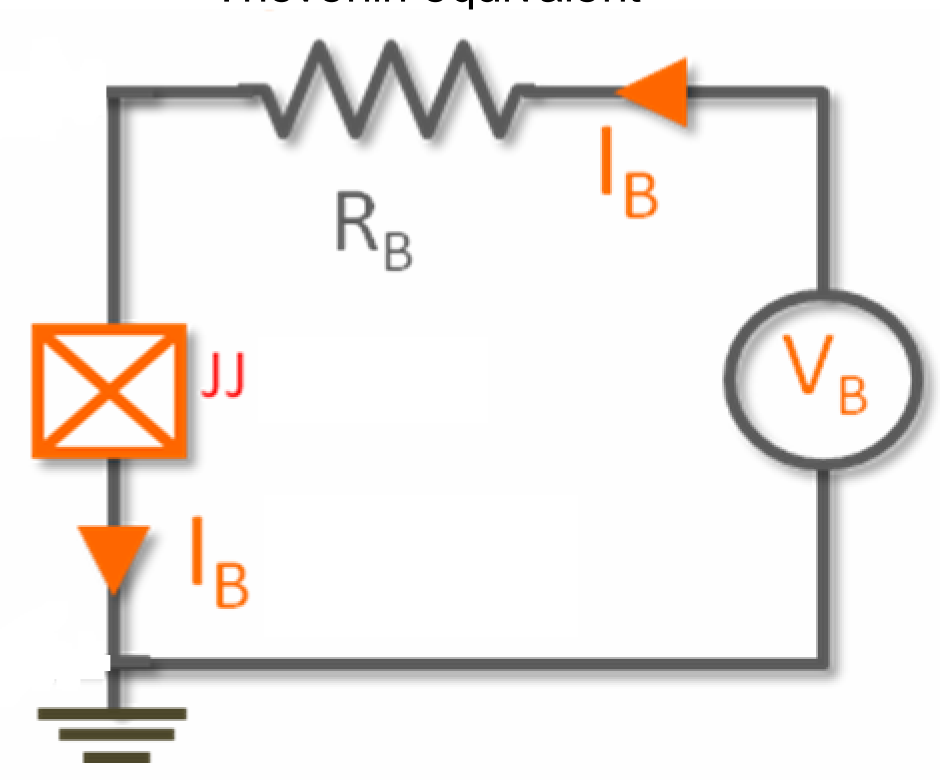
Watch us at:
http://coolphongroup.ifpan.edu.pl

Fridge wiring

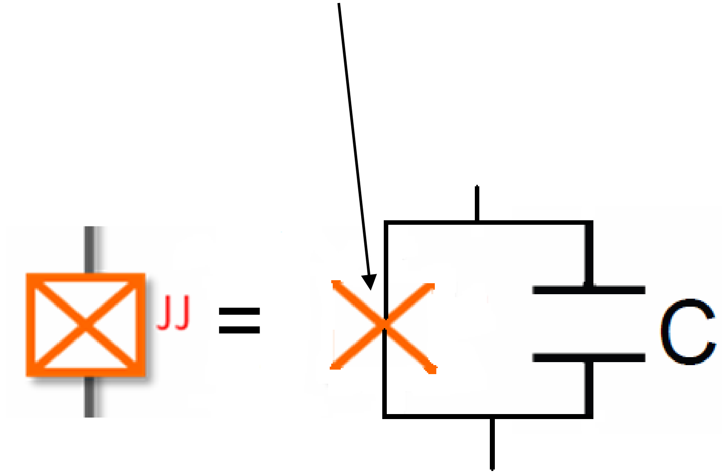


Electrical circuit

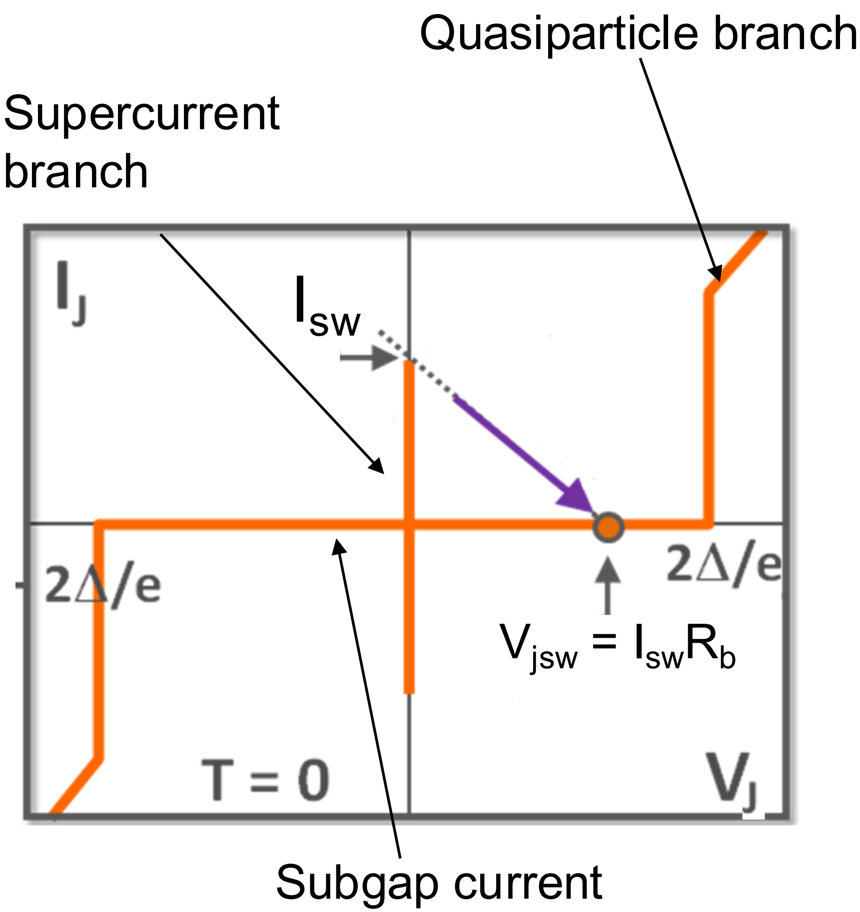
Thevenin equivalent



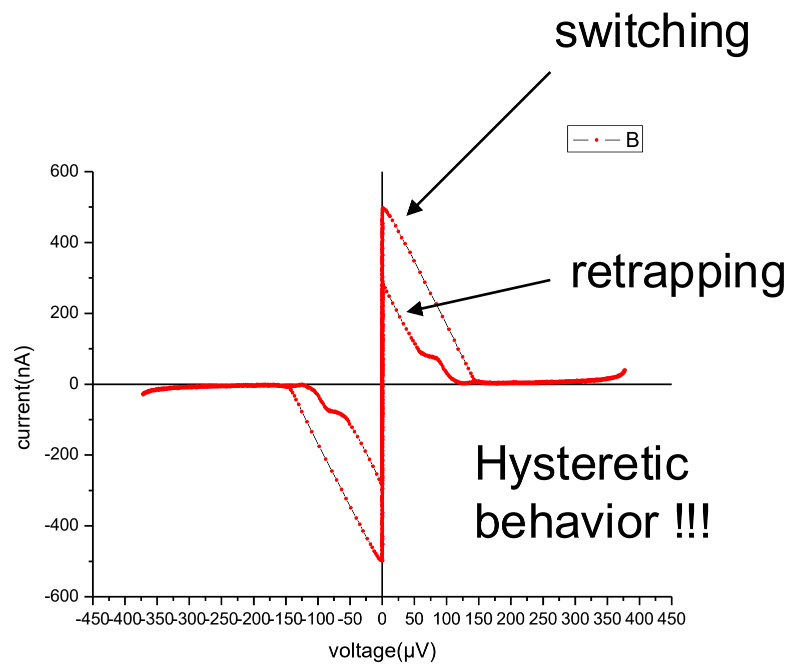
Pure Josephson element
obeying Josephson relations



IV curve



I-V characteristics of JJ biased through R_B bias resistor. JJ supports supercurrent only to certain level. On crossing the threshold value I_0 finite voltage develops across JJ.



$$I_j = I_j(V_j)$$

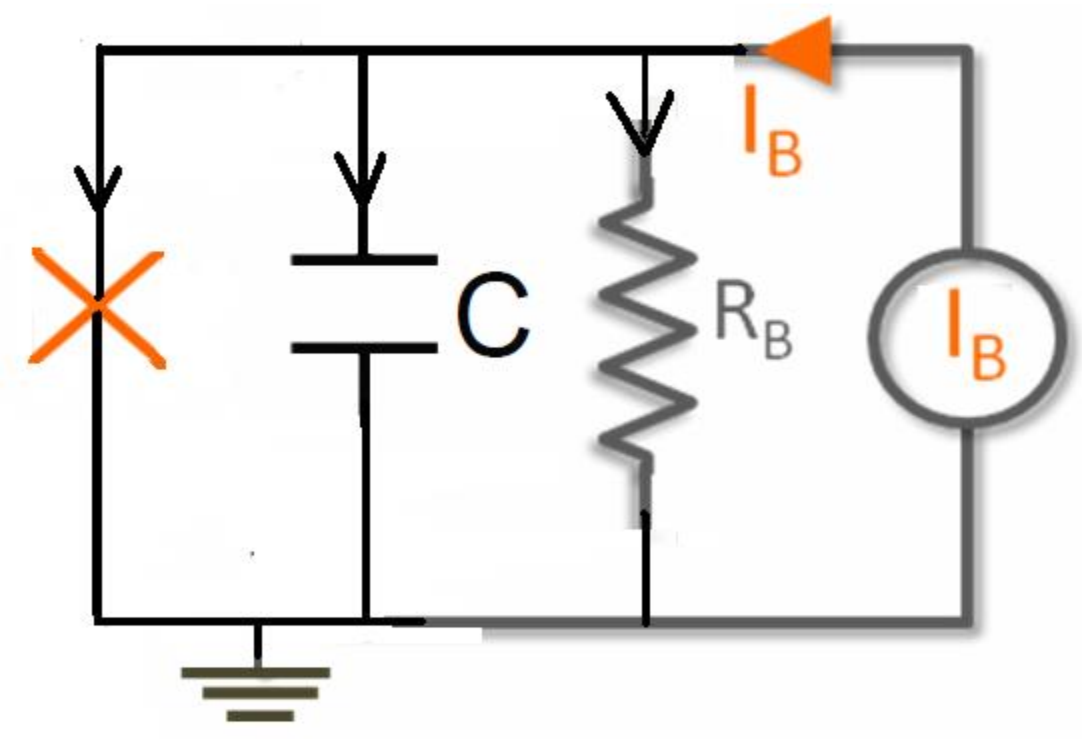
$$V_B = I_j \cdot R_b + V_j \Rightarrow I_j = -\frac{1}{R_b} \cdot V_j + \frac{1}{R_b} \cdot V_b \ (load \ \ line)$$

$$At \ switching \ V_j = 0, V_b = V_{bsw} \ and \ I_j = I_{sw} = \frac{V_{bsw}}{R_b}$$

$$I_j = -\frac{1}{R_b} \cdot V_j + I_{sw} \ (switching \ line)$$

RCSJ model

(Resistively and Capacitively Shunted Junction)



$$I_b = I_R + I_C + I_{JJ} = \frac{V}{R} + C\frac{dV}{dt} + I_0 \cdot \sin\gamma$$

RCSJ model

$$I_b = I_R + I_C + I_{JJ} = \frac{V}{R} + C\frac{dV}{dt} + I_0 \cdot \sin\gamma$$

$$\dot{\gamma} = \frac{1}{\varphi_0} \cdot V$$

$$I_b = \frac{\varphi_0}{R}\dot{\gamma} + C\varphi_0\ddot{\gamma} + I_0 \sin\gamma$$

First, consider γ->0, I_B=0 and map it into harmonic oscillator:

$$C\varphi_0{}^2\ddot{\gamma} + \frac{\varphi_0{}^2}{R}\dot{\gamma} + \varphi_0 I_0 \gamma = 0$$

$$m = C\varphi_0{}^2 \;\; , \;\; \omega_0 = (\frac{I_0}{C \cdot \varphi_0})^{1/2}, b = \frac{\varphi_0{}^2}{R}, Q_0 = \frac{\omega_0}{b/m} = RC\omega_0, k = \varphi_0 I_0 = E_J$$

Back to full equation:

$$\ddot{\gamma} + \frac{\omega_0}{Q_0}\dot{\gamma} + \omega_0{}^2(\sin\gamma - \frac{I_b}{I_0}) = 0$$

Harmonic oscillator:

$$m\ddot{x} + b\dot{x} + kx = 0$$

$$E(t) = E_0 e^{-\gamma t}(energy\ of\ damped\ harmonic\ oscillator)$$

$$\gamma = \frac{b}{m}, \qquad Q = \frac{\omega_0}{\gamma}, \qquad -kx = -\nabla E_p$$

$$\ddot{x} + \frac{\omega_0}{Q}\dot{x} + \omega_0{}^2 x = 0$$

Q - quality factor, amplitude of harmonic oscillator falls by a factor of e in Q/π cycles of free oscillations

$$\ddot{\gamma} + \frac{\omega_0}{Q_0}\dot{\gamma} + \omega_0{}^2(\sin\gamma - \frac{I_b}{I_0}) = 0$$

Wygląda jak harmonic oscillator, ale teraz restoring force wynosi nie kγ (jak w prawie Hooka) tylko:

$$F = -k(\sin\gamma - \frac{I_b}{I_0}) = -\nabla E_p$$

$$E_p = +k\int(\sin\gamma - \frac{I_b}{I_0})d\gamma = -E_J(\cos\gamma + \frac{I_b}{I_0}\gamma) \quad \textit{tilted washboard potential}$$

Tilted washboard potential

$$E_p = -E_J(\cos\gamma + \frac{I_b}{I_0}\gamma)$$

γ <-> x

V/φ_0 (napięcie) <-> v (prędkość)

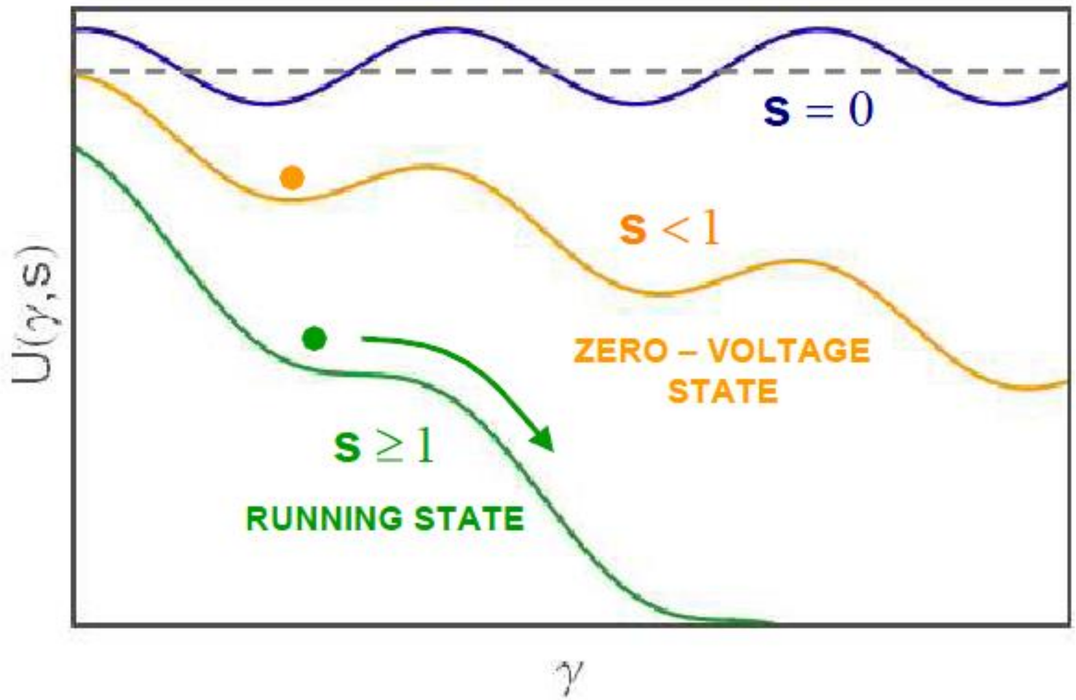


Fig. 2.3. Tilted washboard potential associated to the dynamics of the phase in absence of fluctuations (zero temperature in this classical model). For $s \leq 1$, the potential presents wells in which the particle is trapped. For $s \geq 1$, the wells disappear and the particle gets into to a running state.

Q (quality factor) <-> hysteresis

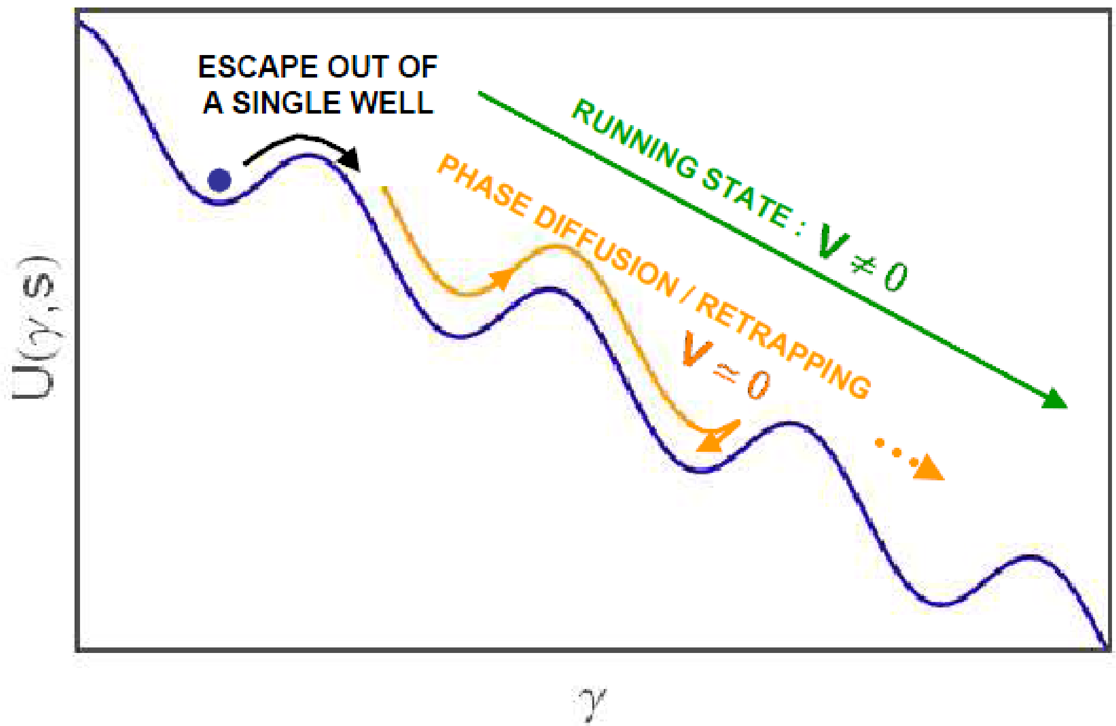


Fig. 2.4. Dynamics of the particle in presence of fluctuations (finite temperature in this classical model). The particle can overcome the barrier and escape out of the single well. The dynamics of the particle depends on the damping. If the damping is small, the particle gains enough energy to reach a running state, leading to a finite voltage. Otherwise, the particle enters a diffusion regime, escaping from one well to be retrapped in a following one, with a very small average velocity.