Superconductivity: macroscopic manifestation of quantum physics

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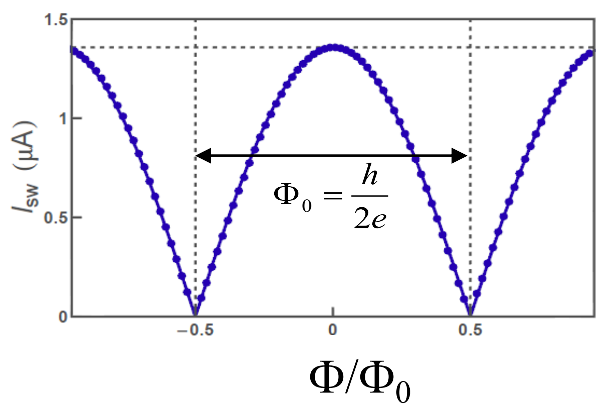
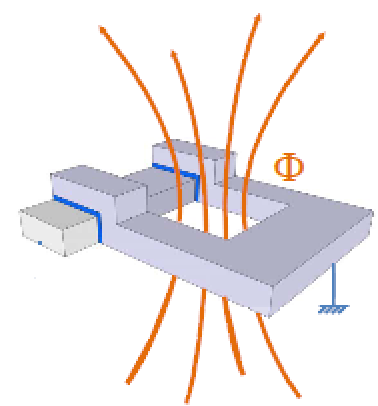
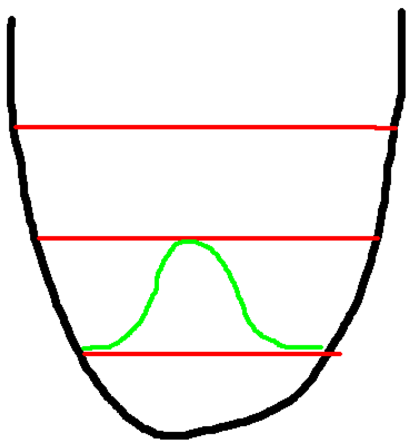
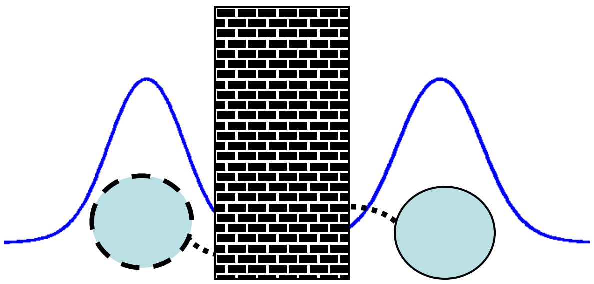




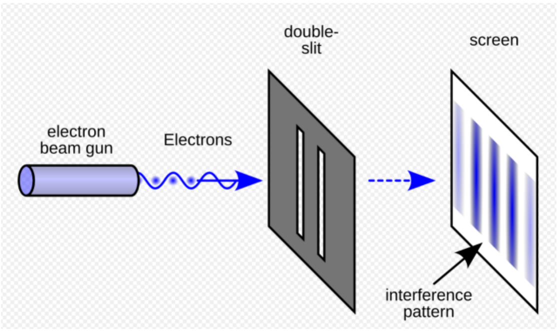
Quantum world



Discrete energy levels



Tunnelling



https://en.wikipedia.org/wiki/Double-slit_experiment#/media/File:Double-slit.svg

Interference/diffraction

Phenomena associated with single atoms/molecules

Can macroscopic objects (ensembles of particles) behave in a quantum way, i.e. as they were a single particle?

Engineered *artificial atoms* consisting of many particles?

A collective state needed…

Well known example: LASER

Conservation of probability

$$\frac{\partial P}{\partial t} = -\nabla \vec{j} = \psi^* \frac{\partial \psi}{\partial t} + \psi \frac{\partial \psi^*}{\partial t}$$

$$\vec{j} = \frac{1}{2}\left\{\left[\frac{p-qA}{m}\psi\right]^* \psi + \psi^* \left[\frac{p-qA}{m}\right]\psi\right\}$$

Macroscopic wavefunction

$$\Psi(r) = \sqrt{\rho} e^{i\theta(r)}$$

$$j_s = \frac{-\hbar}{2m_e}(\nabla\theta - \frac{2e}{\hbar}A)\rho(2e)$$

j_s-supercurrent density

$$\vec{j} = \frac{\hbar}{m}(\nabla\theta - \frac{q}{\hbar}A)\cdot\zeta$$

Formula in Feynman treats ro as a density of charge and m and q as a mass and charge of the Cooper pair respectively

Can we see the phase?

Fluxoid quantization

$$-\nabla\theta = \frac{-m_e j_s}{e\rho\hbar} + \frac{2eA}{\hbar}$$

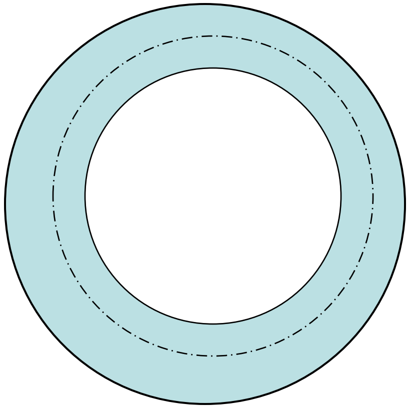
$$\oint -\nabla\theta dl = \oint \frac{-m_e j_s}{e\rho\hbar} dl + \oint \frac{2eA}{\hbar} dl$$

$$2\pi n = \oint \frac{-m_e j_s}{e\rho\hbar} dl + \frac{2e}{\hbar}\Phi_{tot}$$

$$\underbrace{\Phi_{tot} - \frac{m_e}{2e^2\rho}\oint j_s dl}_{\text{fluxoid}} \equiv \Phi' = n\frac{h}{2e} = n\,\Phi_0$$

LONDON, 1950

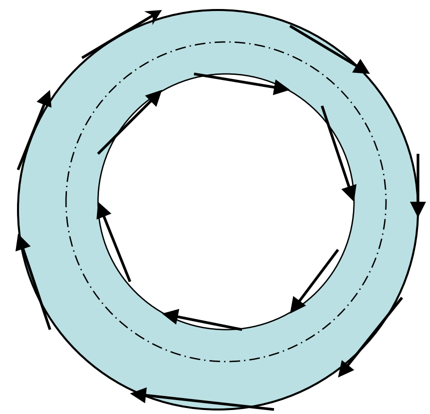
$\Phi_0 = \frac{h}{2e}$ fluxon (flux quantum)



Superconducting Ring

j = 0 inside the ring due to the Meissner effect

screening current I_s flows only on the surface of the ring

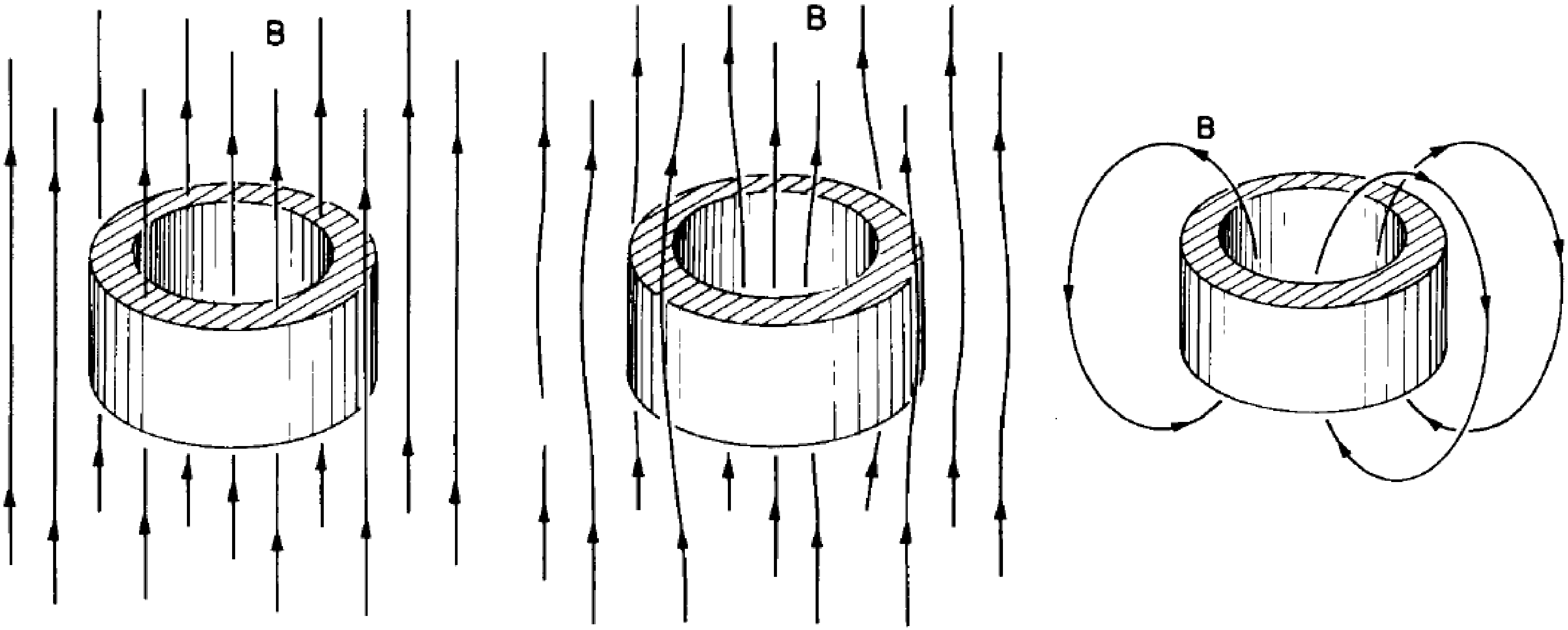


$$\Phi_{tot} = n\,\Phi_0$$

$$\Phi_{tot} = \Phi_{ext} - L_{geom} I_s$$

Fluxoid quantization becomes flux quantization !!!

Superconducting Ring – Meissner effect



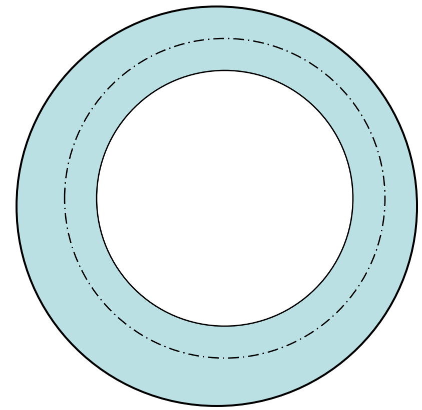
$$\Phi_{tot} = n\,\Phi_0$$

$\Phi_0 = \frac{h}{2e}$ fluxon (flux quantum)

What happens if j ≠ 0?

$$-\nabla\theta = \frac{-m_e j_s}{e\rho\hbar} + \frac{2eA}{\hbar}$$

$$\oint -\nabla\theta dl = \oint \frac{-m_e j_s}{e\rho\hbar} dl + \oint \frac{2eA}{\hbar} dl$$



$$2\pi n = \oint \frac{-m_e j_s}{e\rho\hbar} dl + \frac{2e}{\hbar}\Phi_{tot}$$

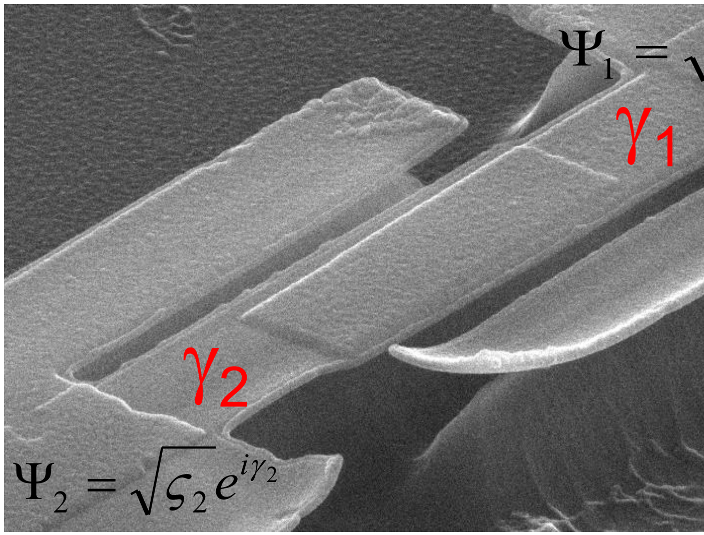
$$\underbrace{\Phi_{tot} - \frac{m_e}{2e^2\rho}\oint j_s dl}_{\text{fluxoid}} \equiv \Phi' = n\frac{h}{2e} = n\,\Phi_0$$

LONDON, 1950

$\Phi_0 = \frac{h}{2e}$ fluxon (flux quantum)

$$\boxed{\begin{array}{ll} \Phi_{tot} - L_K I = n\,\Phi_0 & \Phi_{tot} = \Phi_{ext} - L_{geom} I \\ \Phi_{ext} - (L_K + L_{geom})I = n\,\Phi_0 & \end{array}}$$

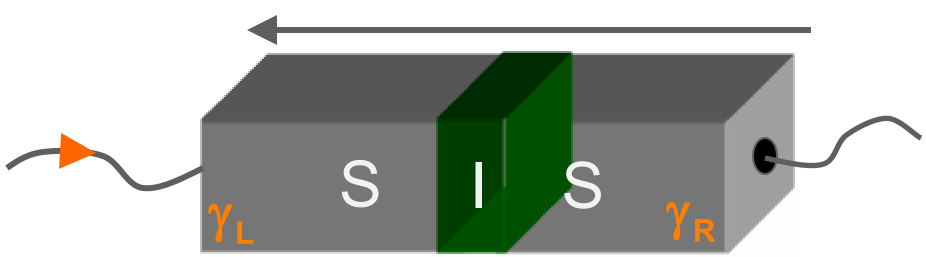
Josephson relations



$$I = I_0 \cdot \sin(\gamma_2 - \gamma_1)$$

$$\frac{d(\gamma_2 - \gamma_1)}{dt} = \frac{2 \cdot e}{\hbar} \cdot V$$

$$V = (\hbar / 2e)\dot{\gamma}$$



$$I_J = I_0 \sin \gamma$$

Derivation of Feynman relations:

Feynman, vol. III, seminar on superconductivity

E-beam lithography

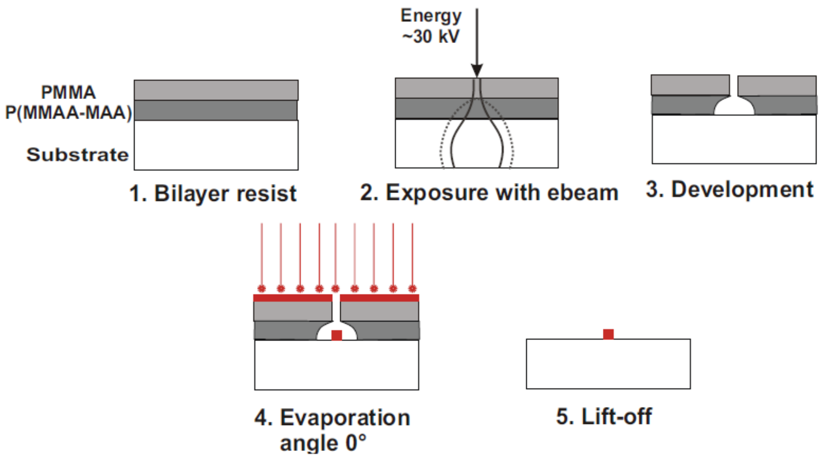


Figure 2.1 Fabrication of metallic structures by using positive electron beam lithography and evaporation techniques. Bilayer resist is used to achieve the undercut structure shown in step 3.

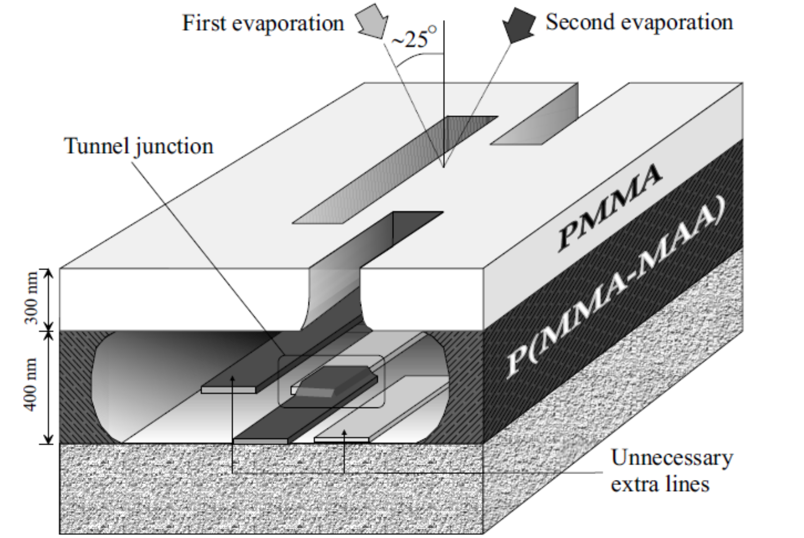
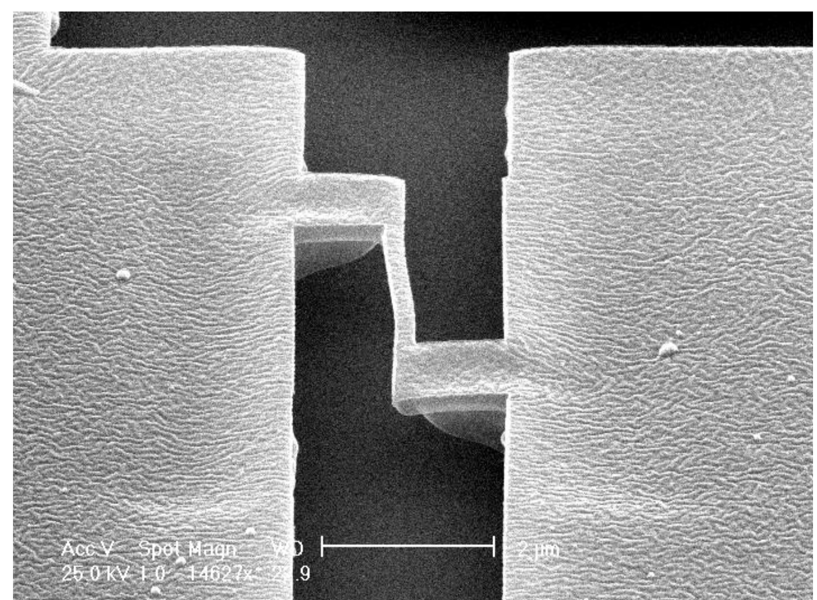


FIGURE 2.1 Principle of the self-aligning shadow evaporation technique. The tunnel junction is formed between the two metallic layers (usually Al) evaporated at different angles so that they overlap slightly. The tunnelling barrier is formed by oxidising the first layer before evaporating the second one. As a side effect of the technique there will be some extra lines, which are not used in measurements.



Photolithography

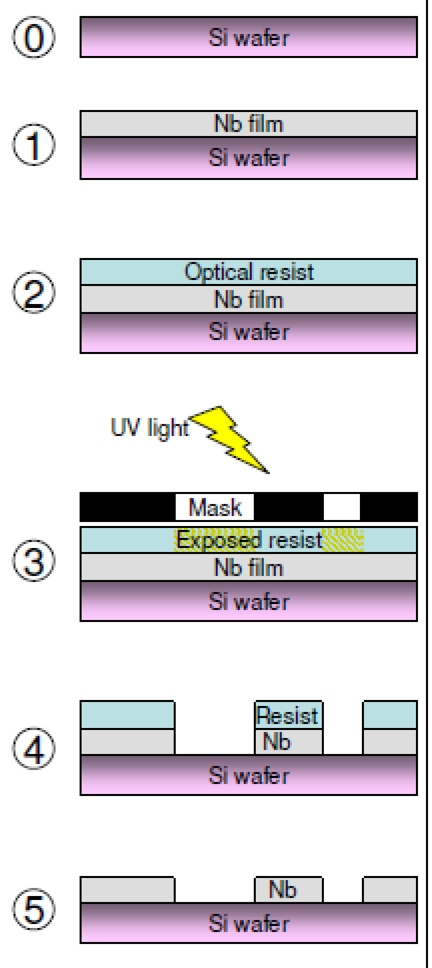
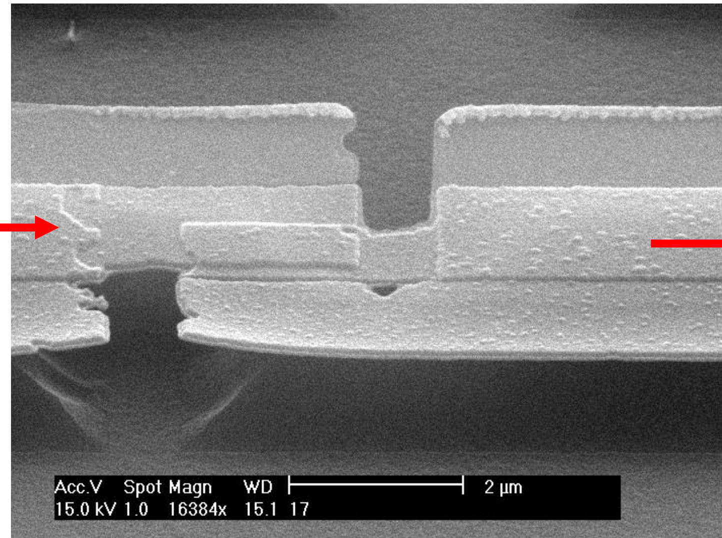
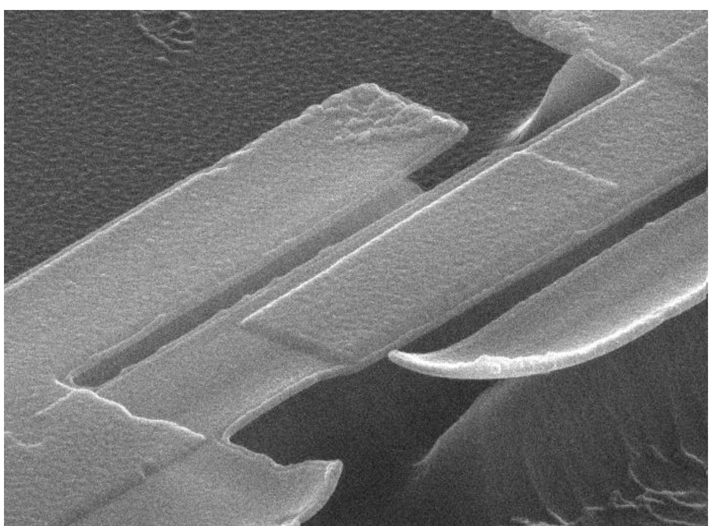


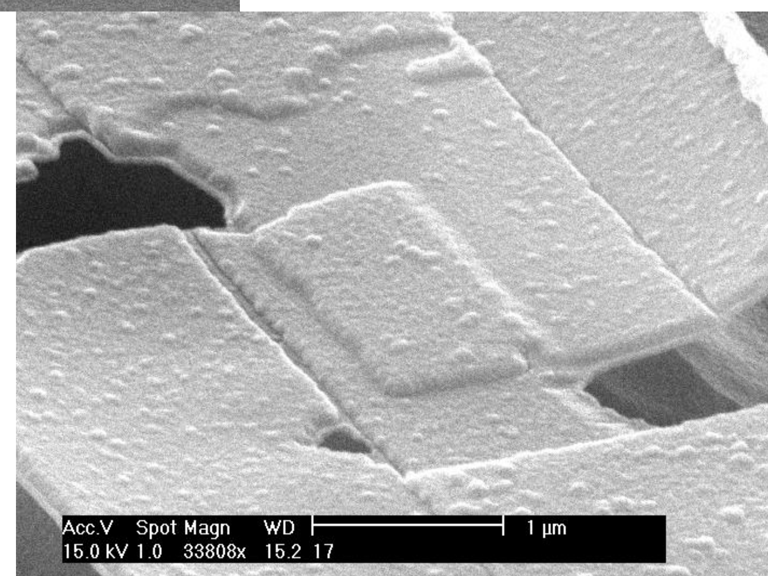
Figure 2.10: *Optical lithography process*

Tunnelling Josephson junction (JJ)

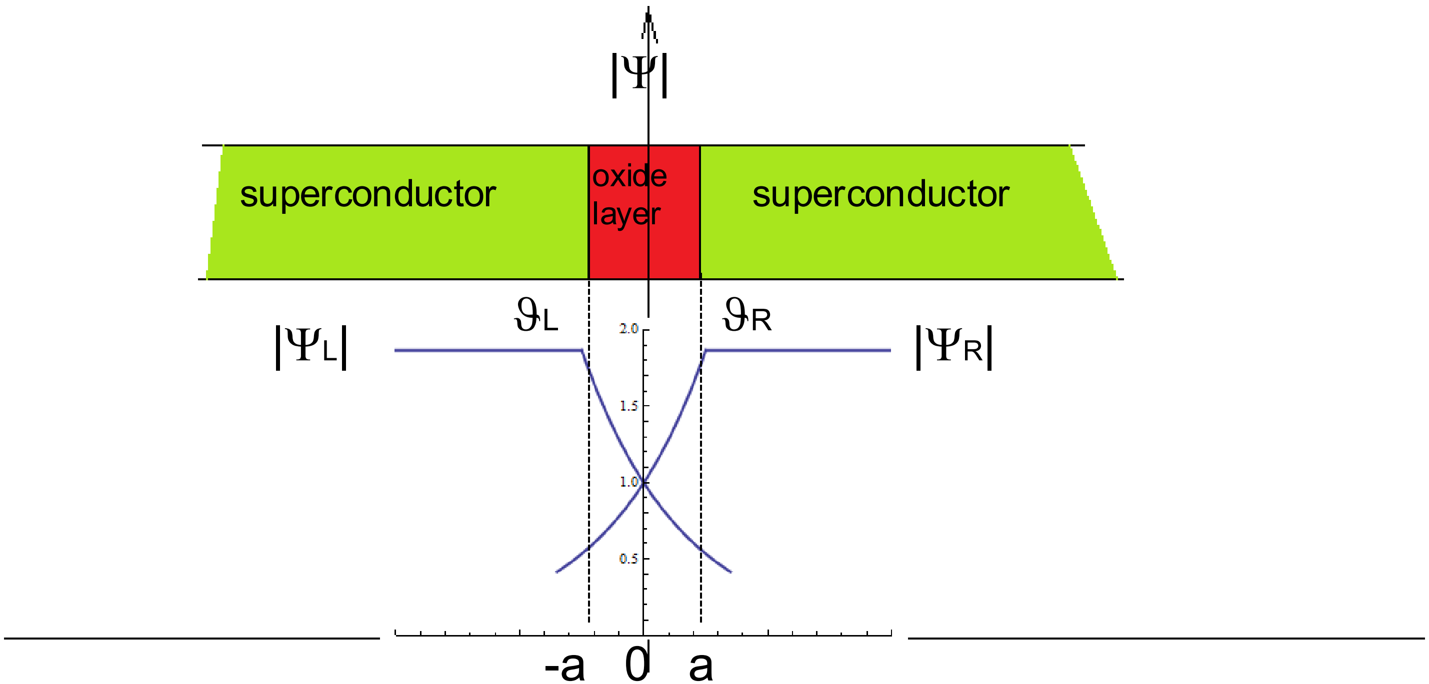
current







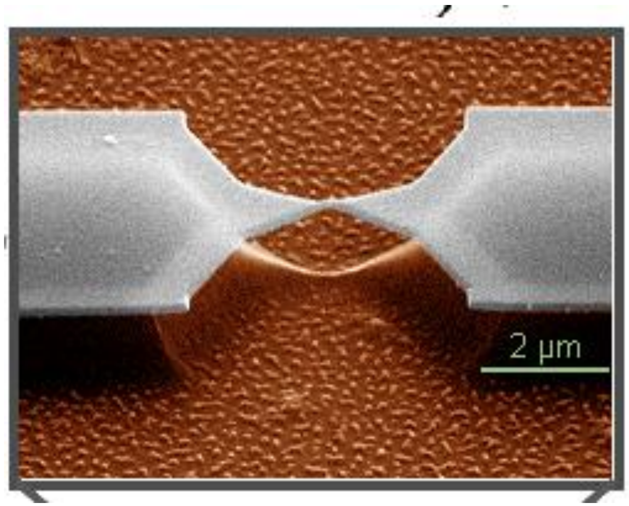
Tunnelling Josephson junction (JJ)

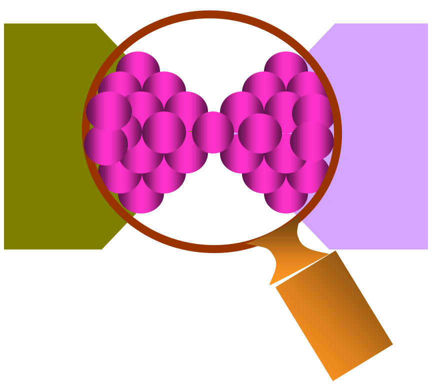


$$\psi = \sqrt{\varsigma}\exp(i\vartheta_L)\exp(-\frac{l+a}{\lambda}) + \sqrt{\varsigma}\exp(i\vartheta_R)\exp(\frac{l-a}{\lambda})$$

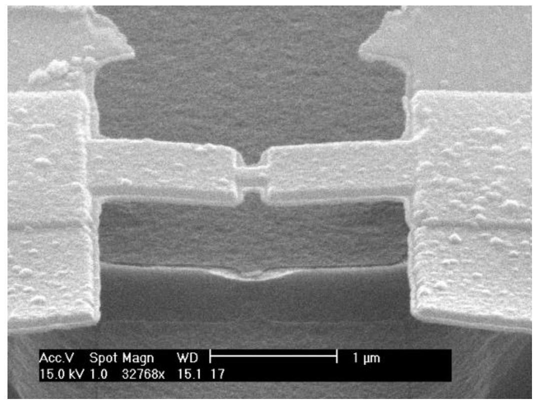
ZOO of Josephson junctions

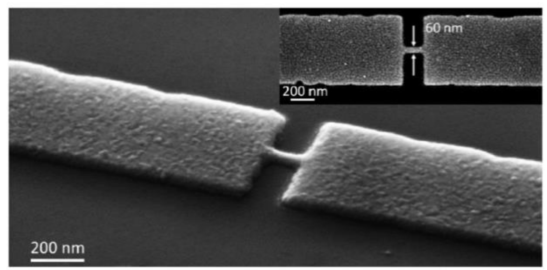
Atomic point contact

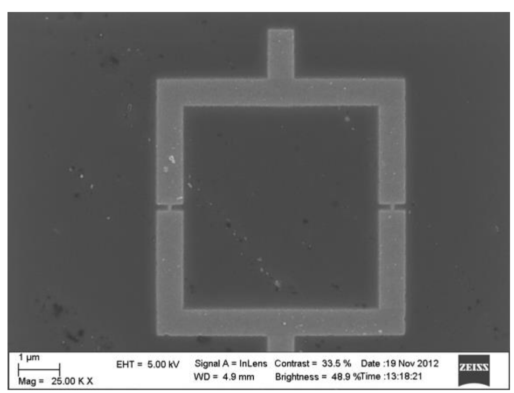




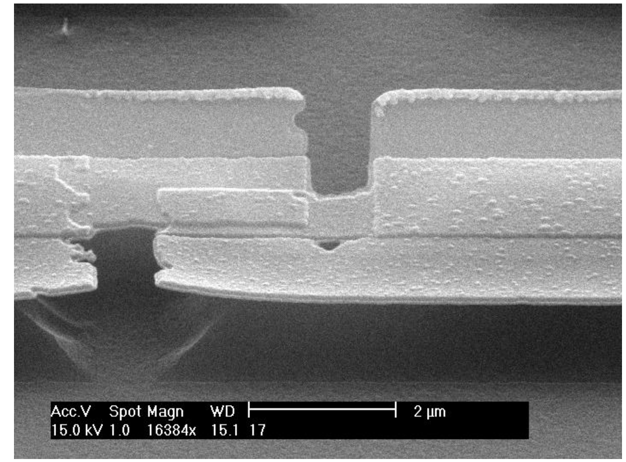
Diffusive weak link







Tunnel junction



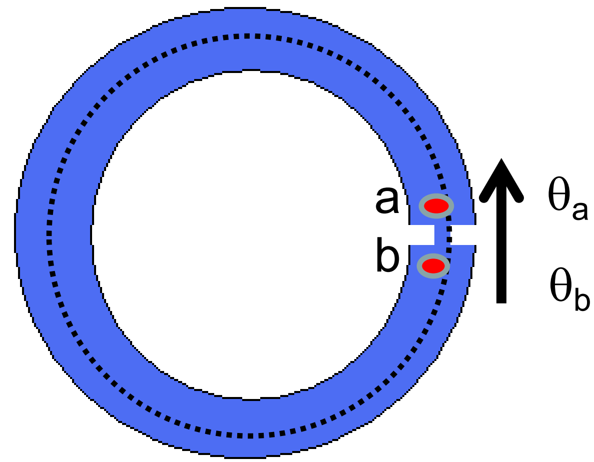
Note: The famous 1st Josephson relation holds only for tunnelling junctions.

But it can be extended to treat all types of JJ !!!

⇒ We will need
ANDREEV BOUND STATES
(our next lecture)

Ring + 1 Josephson junction

$$2\pi n = \oint \frac{-m_e j_s}{e\rho\hbar} dl + \frac{2e}{\hbar}\Phi_{tot}$$



$$2\pi n = -\underbrace{\oint_a^b \frac{m_e j_s}{e\rho\hbar} dl}_{\gamma} - \oint_b^a \frac{m_e j_s}{e\rho\hbar} dl + \frac{2e}{\hbar}\Phi_{tot}$$

γ - superconducting phase across a junction

$$\gamma = \theta_a - \theta_b$$

$$2\pi n = \frac{2e}{\hbar}\Phi_{tot} - \gamma - \oint_b^a \frac{m_e j_s}{e\rho\hbar} dl$$

$$(\gamma + 2\pi n) mod(2\pi) = \gamma = \frac{2\pi}{\Phi_0}(\Phi_{tot} - L_K I_s)$$

Phase relation for a ring with a single junction

$$n\Phi_0 = \Phi_{tot} - L_K I_s(\gamma) - \frac{\Phi_0}{2\pi}\gamma$$

Fluxoid quantization for a ring Interrupted with a junction

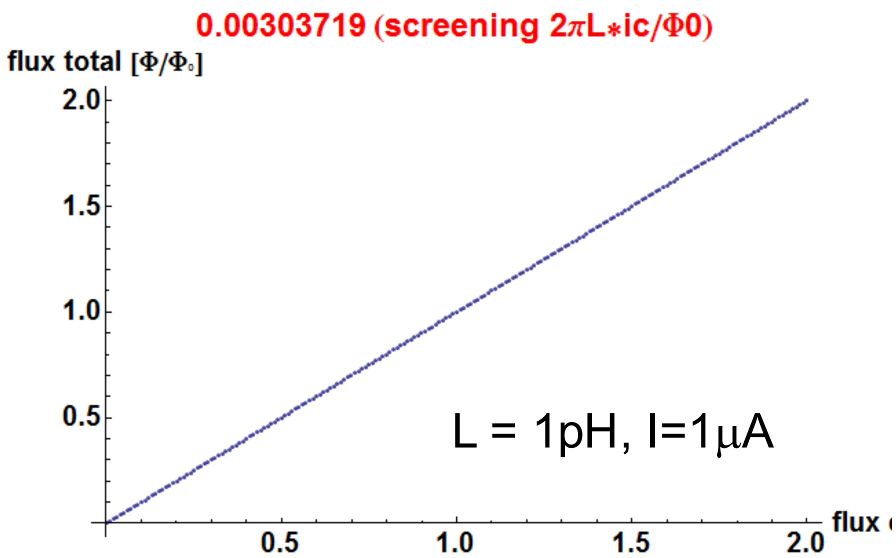
The total flux threading the ring is not quantized any more !!!

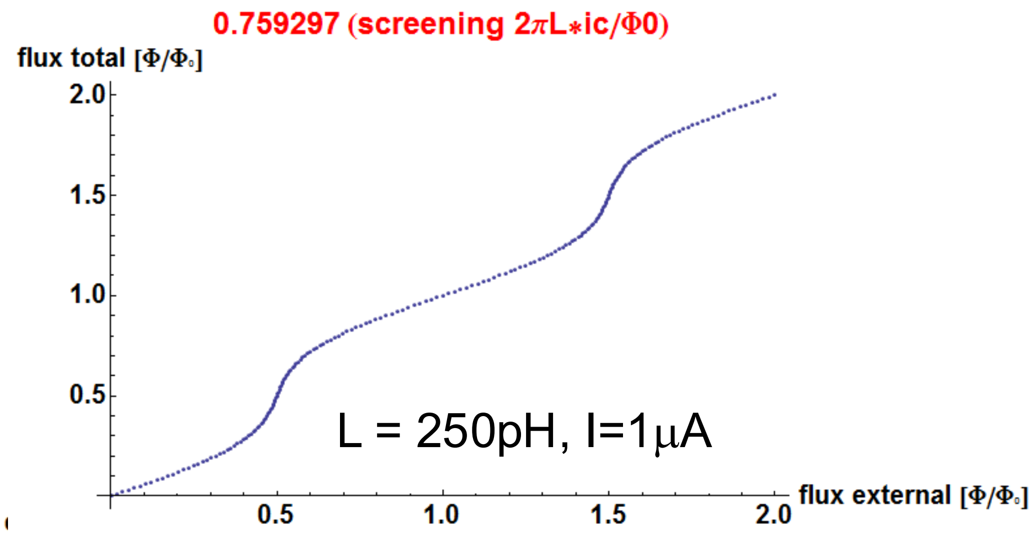
Screening current

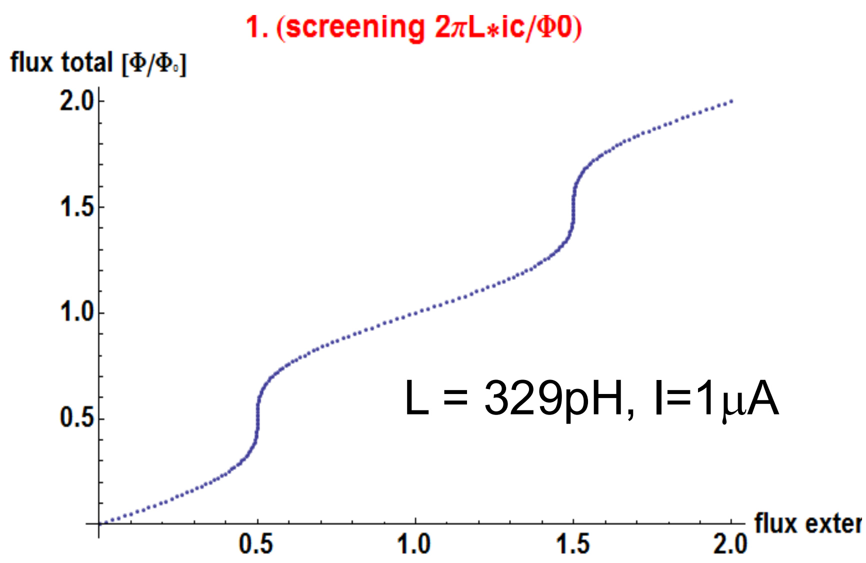
$$\Phi = \Phi_{ext} - (L_{geom}+L_K)I_s = \Phi_{tot} - L_K I_s$$

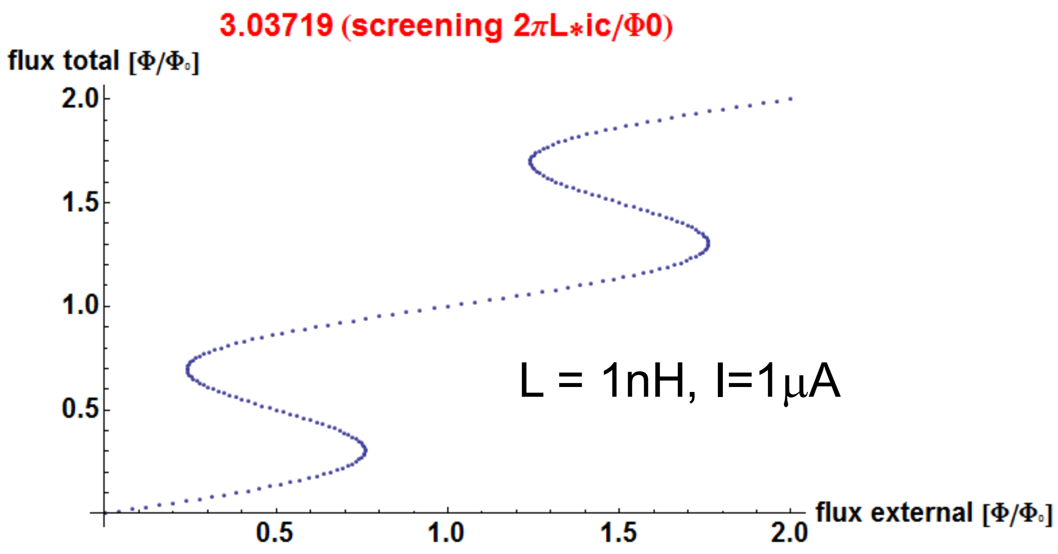
$$\Phi = \Phi_{ext} - LI_c sin(\frac{2\pi}{\Phi_0}\Phi)$$

From weak to strong link…

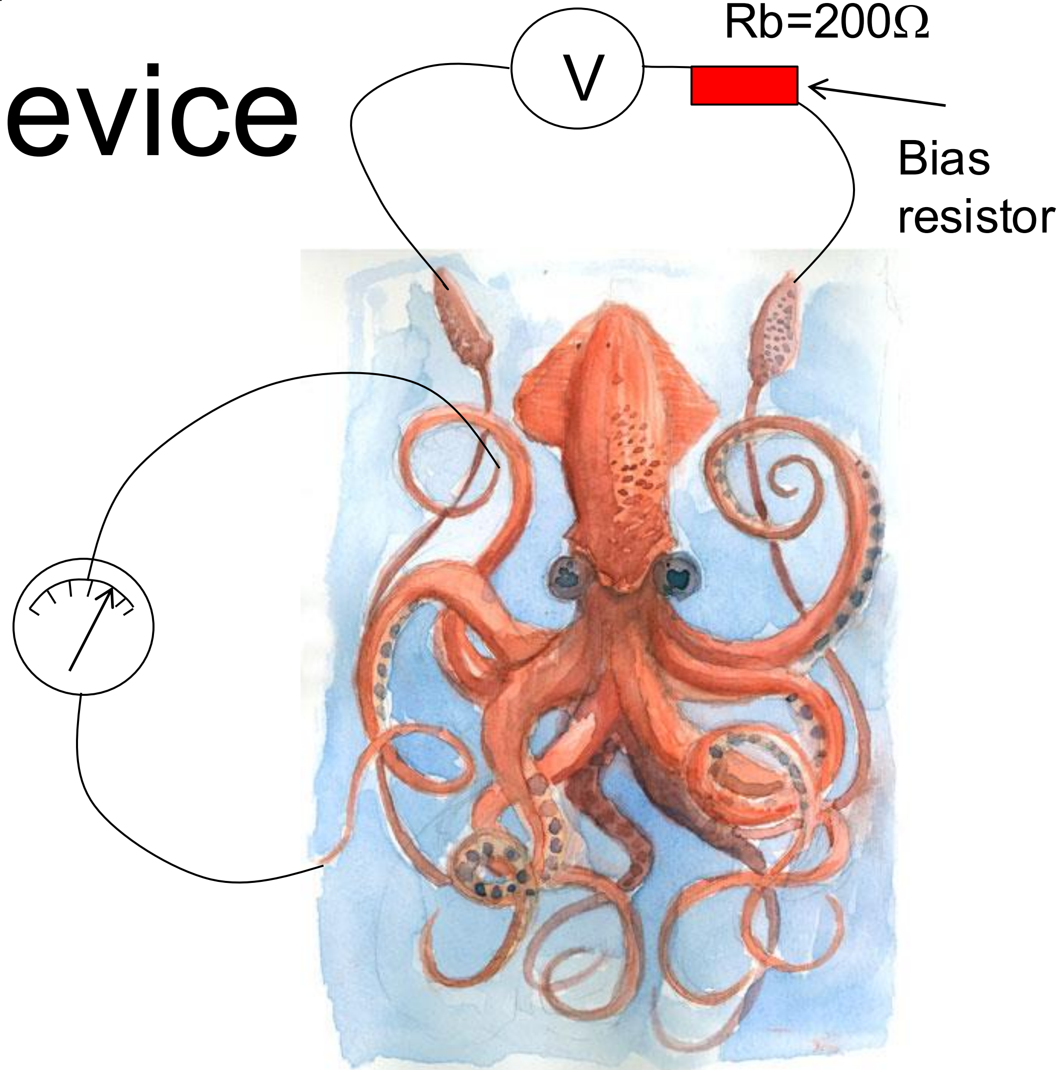


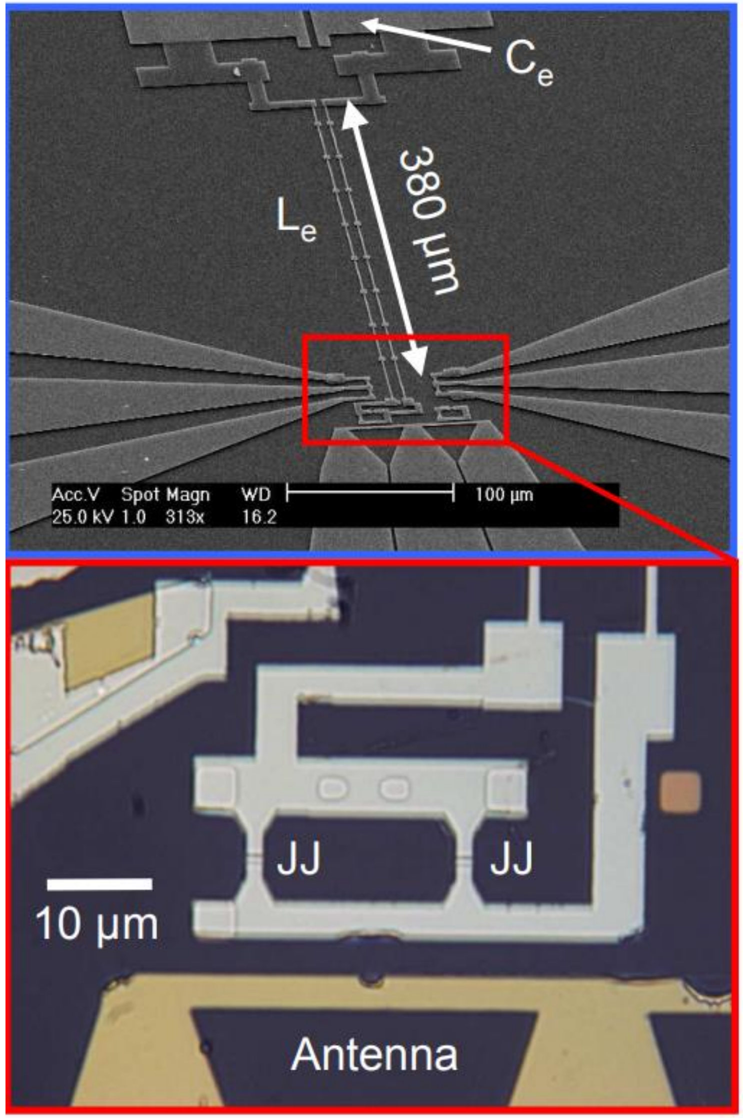






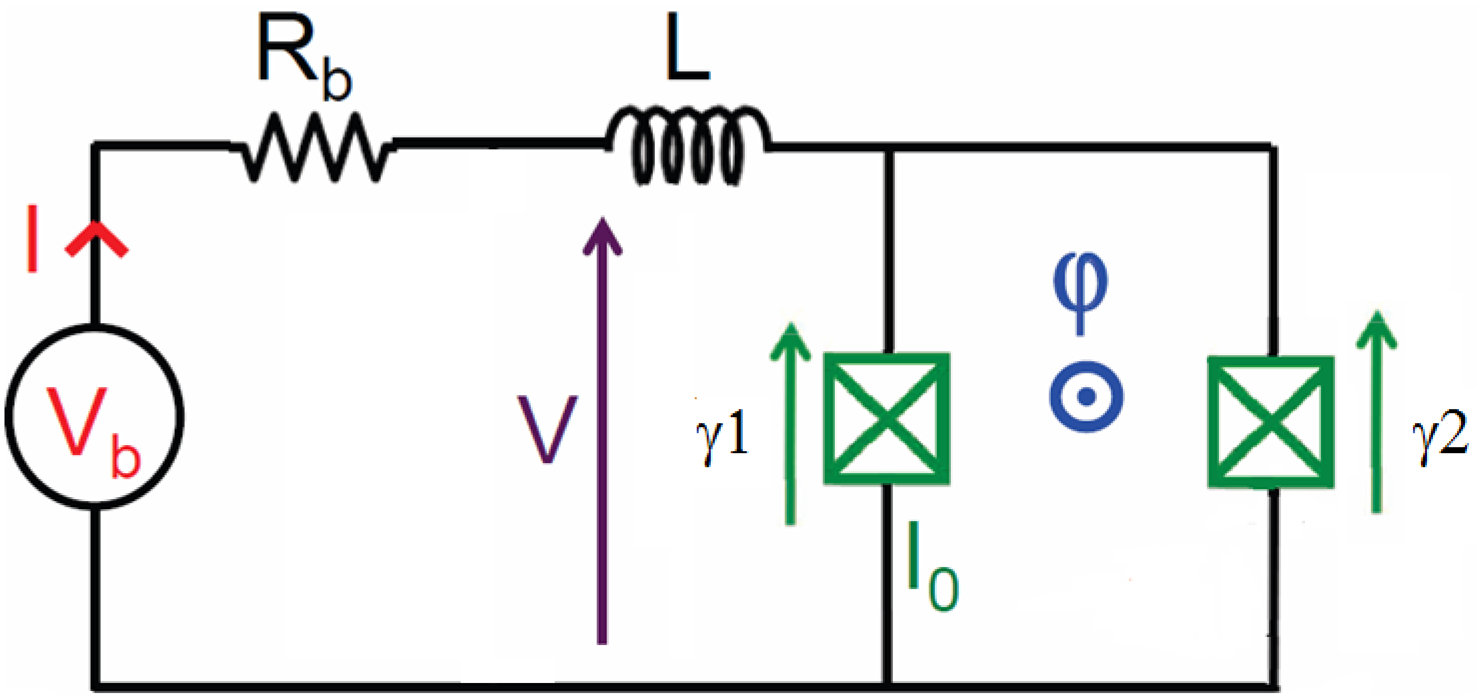
Superconducting Quantum Interference Device (SQUID)





C_e
L_e
380 µm
Acc.V Spot Magn WD 100 µm
25.0 kV 1.0 313x 16.2
JJ
JJ
10 µm
Antenna

SQUID circuit



$$\gamma_1 - \gamma_2 = 2\pi \frac{\Phi}{\Phi_0} = \varphi$$

Φ– strumień magnetyczny przez pętle SQUIDu,

φ - faza „magnetyczna”

Critical current of the SQUID

$$I_{J2} = I_b/2 - I_p(\varphi)$$
$$I_{J1} = I_b/2 + I_p(\varphi)$$

$$I_b = I_{J1} + I_{J2} = I_0(\sin\gamma_1 + \sin\gamma_2) = 2I_0 \sin\frac{\gamma_1+\gamma_2}{2}\cos\frac{\gamma_1-\gamma_2}{2}$$

$$I_b = 2I_0 \cos\frac{\varphi}{2}\sin\frac{\gamma_1+\gamma_2}{2}$$

$$I_c = \max I_b \quad => \frac{\gamma_1+\gamma_2}{2} = \frac{\pi}{2} \;\; \& \;\; \gamma_1 - \gamma_2 = \varphi = 2\pi\frac{\Phi}{\Phi_0}$$

$$=> I_c = 2I_0\left|\cos(\pi\frac{\Phi}{\Phi_0})\right| \quad if \;\; \gamma_1 = \frac{\pi}{2} - \frac{\varphi}{2} \;\; \& \;\; \gamma_2 = \frac{\pi}{2} + \frac{\varphi}{2}$$

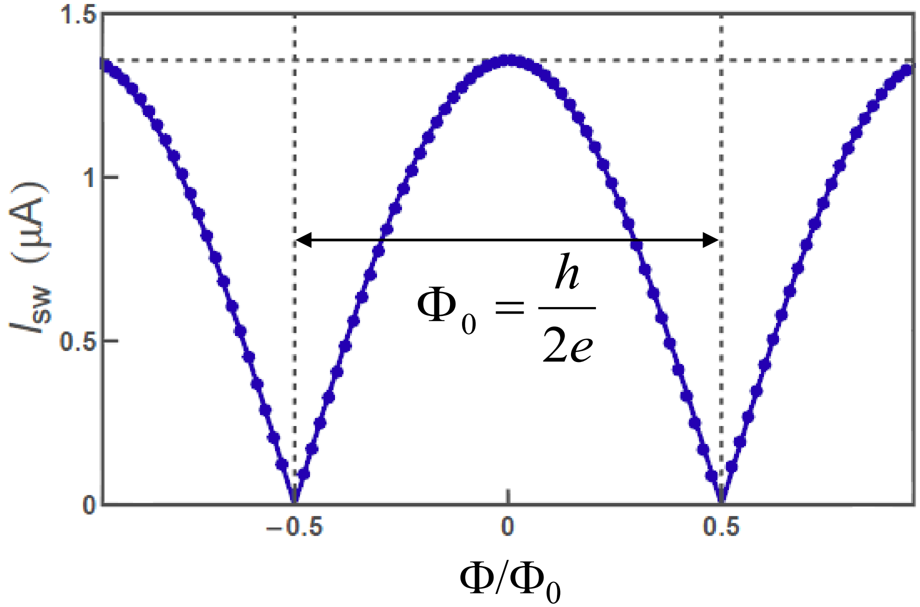
$$\Phi_0 = \frac{h}{2e} = 2\cdot 10^{-15} Wb \; - \; flux \;\; quantum$$

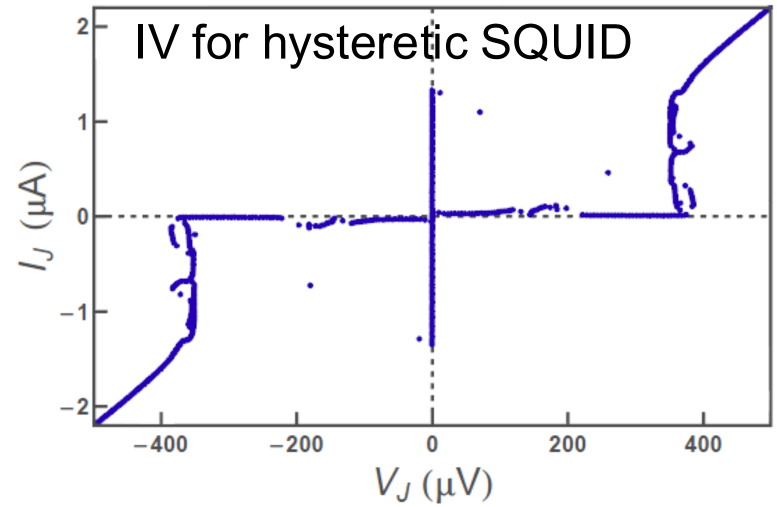
For $\Phi = \Phi_0/2 \;\; => \;\; \gamma_1 = 0, \;\; \gamma_2 = \pi \;\; =>$ $I_{j1} = 0$, $I_{j2} = 0$ => perfect cancellation of currents flowing through two arms of the SQUID

SQUID is a JJ with the magnetic field-regulated critical current

(tunable JJ !!!)

Interference pattern for SQUID





Symmetric Squid is superconducting analog of double slit optical (or electron) interferometer:

applied flux – Φ ⇔ d*sinθ - path difference
Flux quantum – Φ_0 ⇔ λ – wavelength

For symmetric SQUID (2 x JJ): $$I_c = 2 \cdot I_{JJ}^0 \cdot \left|\cos(\pi \cdot \frac{\Phi}{\Phi_0})\right|$$

What will happen for $\Phi = \Phi_0/2$?

Analyze sinusoid graphically to show evolution of two junction phases both with magnetic flux and bias current:

Magnetic flux alone shifts the phases symetrically around zero:
gamma1 = - gamma2, gamma1-gamma2=2pi Fi/Fi0

Bias current shifts two phases as a unisom preserving distance between them gamma1-gamma2=2pi Fi/Fi0

Special case:

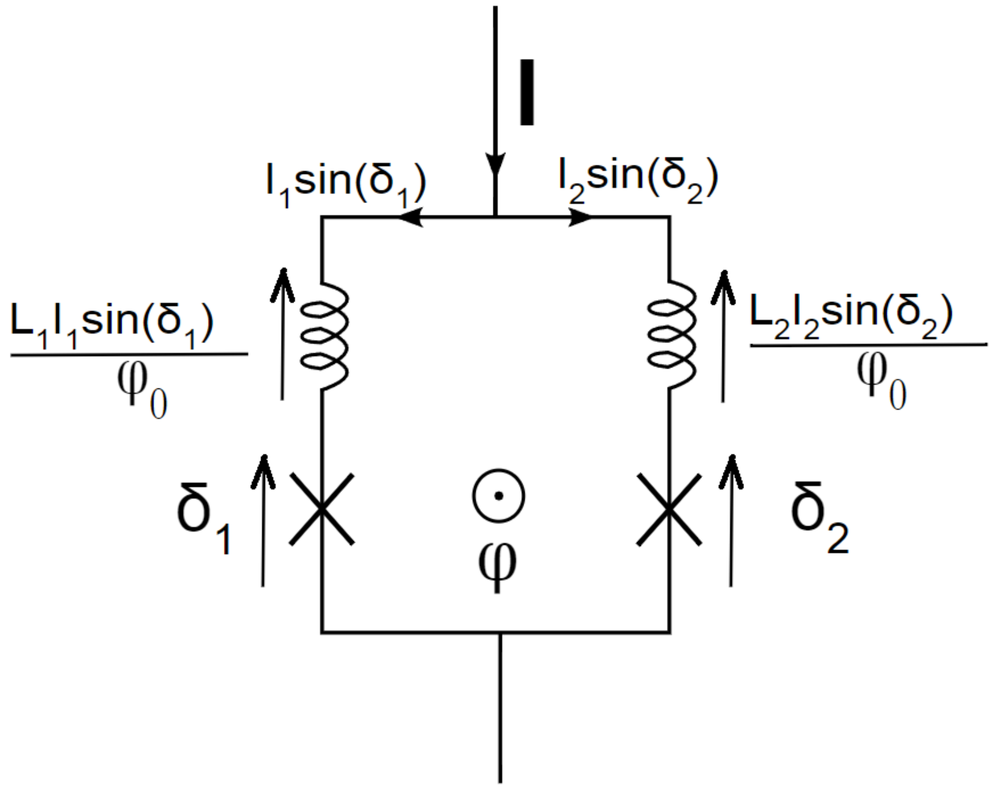
Gamma1-gamma2=Pi, maximum screening current circulating in the SQUID, Equal to the critical current of the single junction I0.

For $\Phi = \Phi_0/2$ => $|\gamma_1 - \gamma_2| = \pi$ => $I_{j1} = -I_{j2} = \mathrm{Max}(I_{JJ})$ => perfect cancellation of currents flowing through two arms of the SQUID

SQUID can not support superconducting current!!!

SQUID – numerical study of a general case

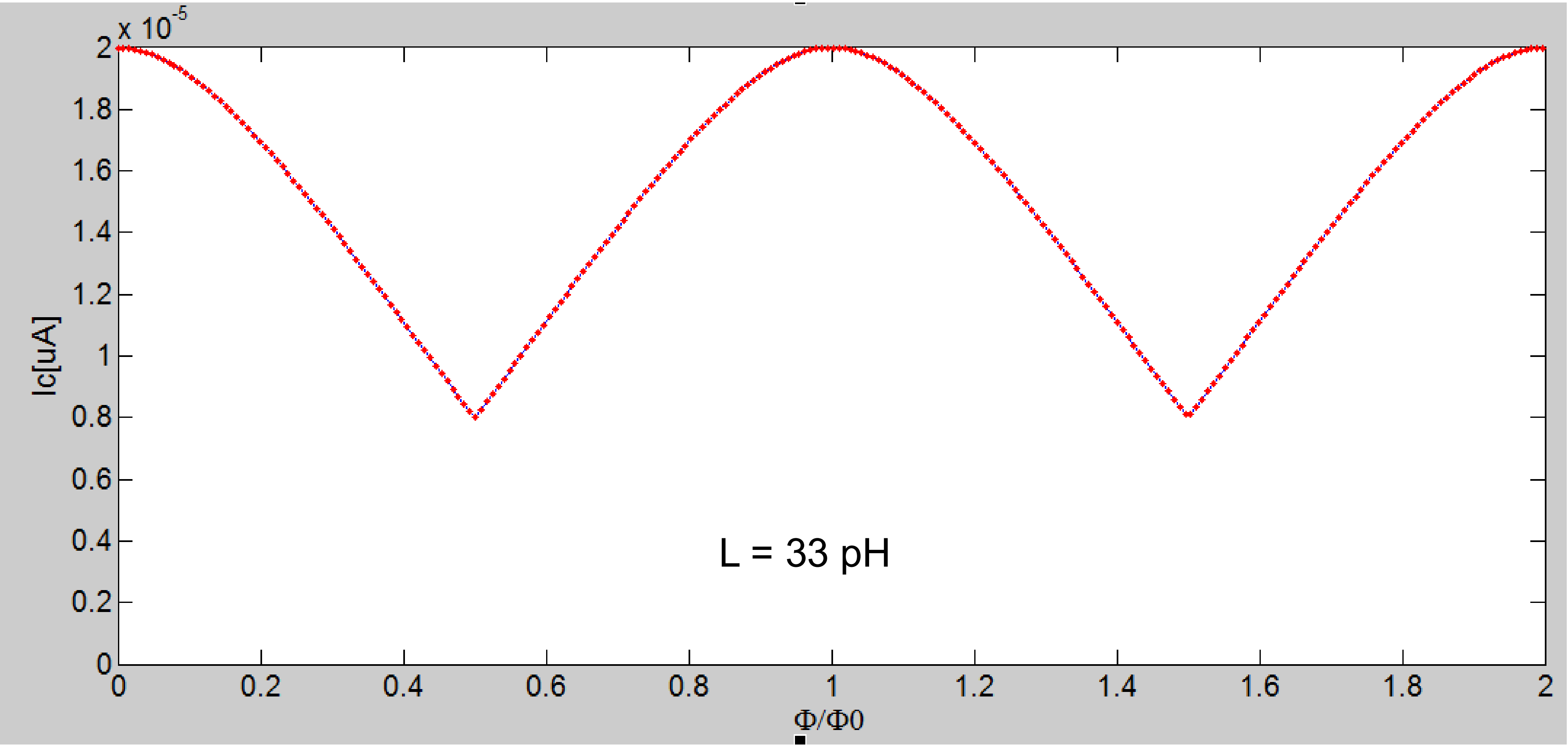
$$L_1 = L(1-\eta)$$
$$L_2 = L(1+\eta)$$
$$I_1 = I_0(1-\alpha)$$
$$I_2 = I_0(1+\alpha)$$



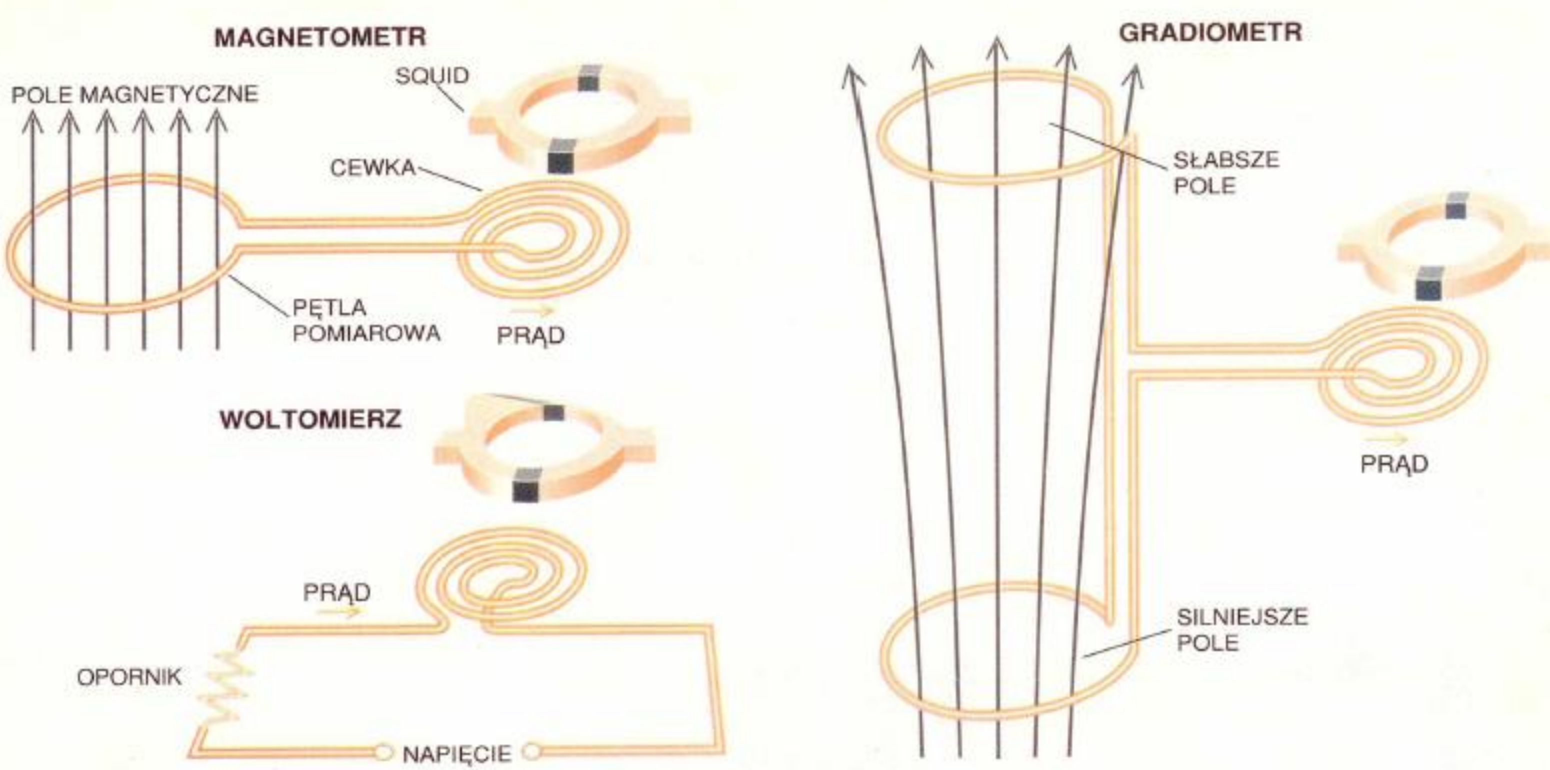
Phase equation: $$\delta_1 + \frac{L_1 I_1 \sin(\delta_1)}{\varphi_0} - \delta_2 - \frac{L_2 I_2 \sin(\delta_2)}{\varphi_0} = 2\pi \frac{\Phi_{ext}}{\Phi_0}$$

SQUID with self-flux

$$\Phi = \Phi_{ext} + LJ_s$$

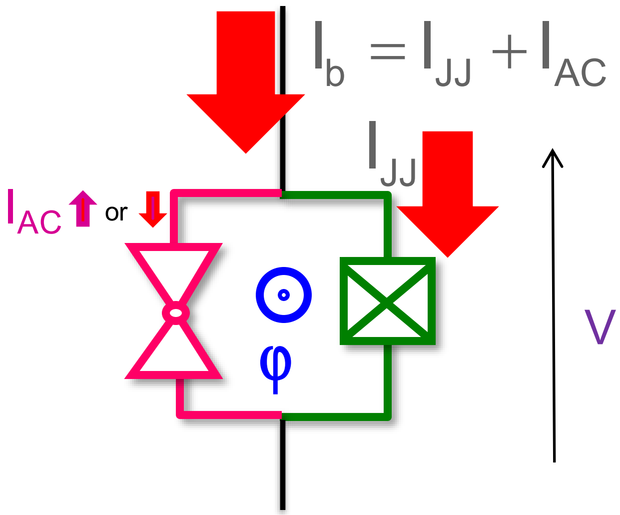


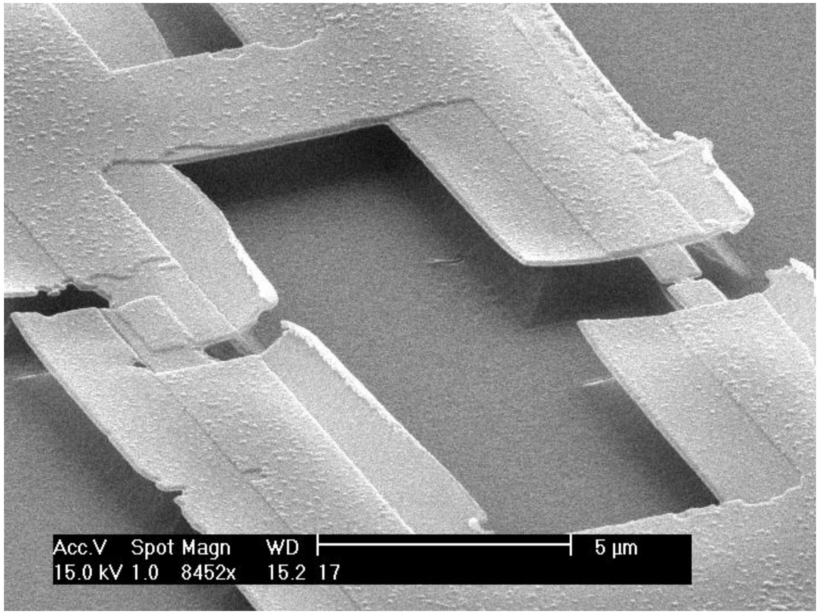
SQUID – various applications



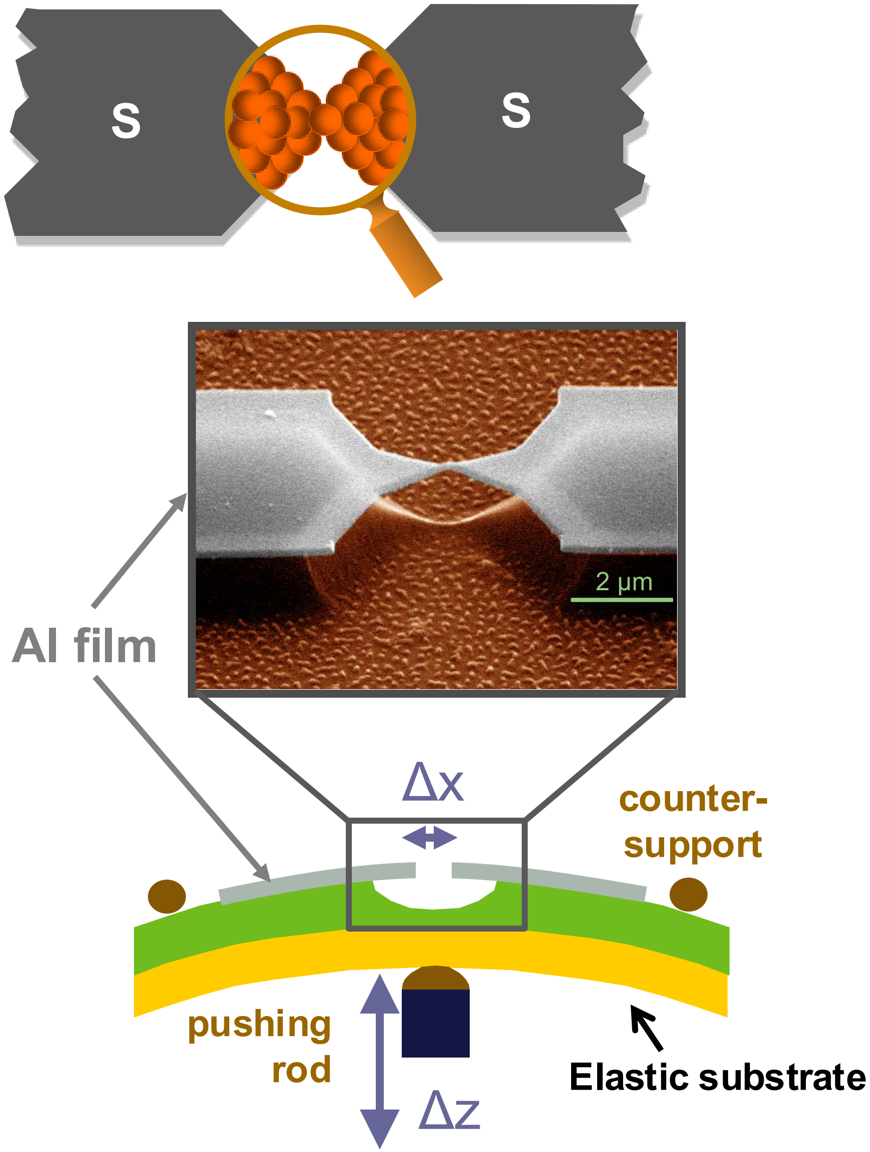
URZĄDZENIA WYKORZYSTUJĄCE SQUID zwykle wymagają dodatkowych elementów. Magnetometr zawiera „transformator strumienia", który składa się z pętli pomiarowej połączonej z cewką sprzężoną z interferometrem. Pole magnetyczne wzbudza prąd w pętli, powodując jego przepływ przez cewkę i powstanie strumienia pola magnetycznego przenikającego interferometr. W gradiometrze dwie oddalone od siebie pętle pomiarowe nawinięte w przeciwnych kierunkach są poddawane jednoczesnemu działaniu pola magnetycznego. Strumień pola magnetycznego przenika interferometr tylko wtedy, gdy natężenie tego pola w obu pętlach jest różne. W woltomierzu mierzone napięcie wywołuje przepływ prądu o natężeniu równym ilorazowi wartości tego napięcia i wielkości oporu połączonego z cewką sprzężoną z nadprzewodnikowym interferometrem kwantowym.

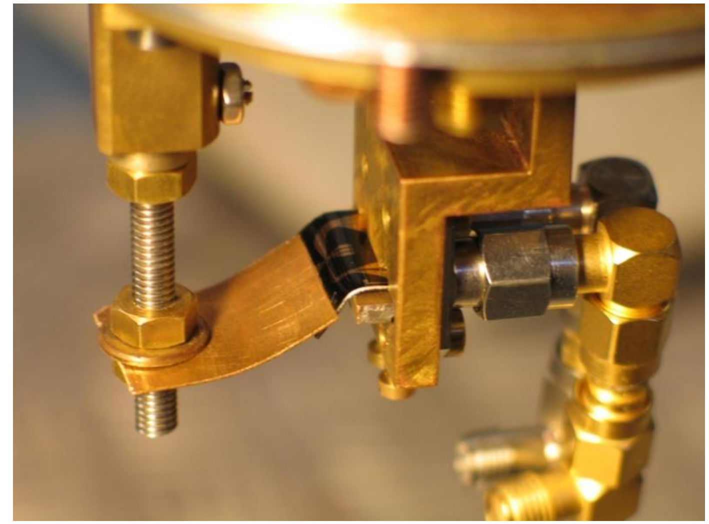
Atomic Squid…- my PostDoc project



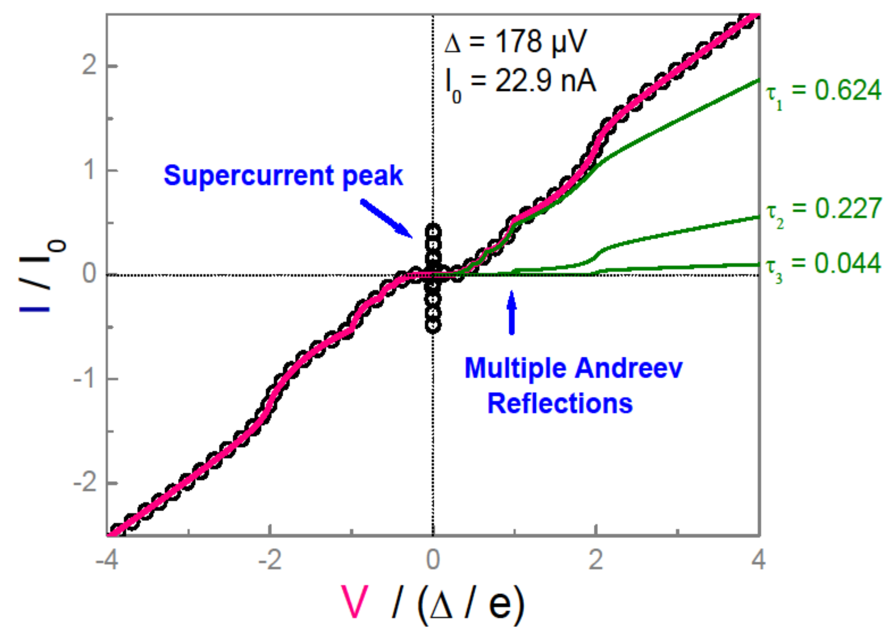


Atomic contact

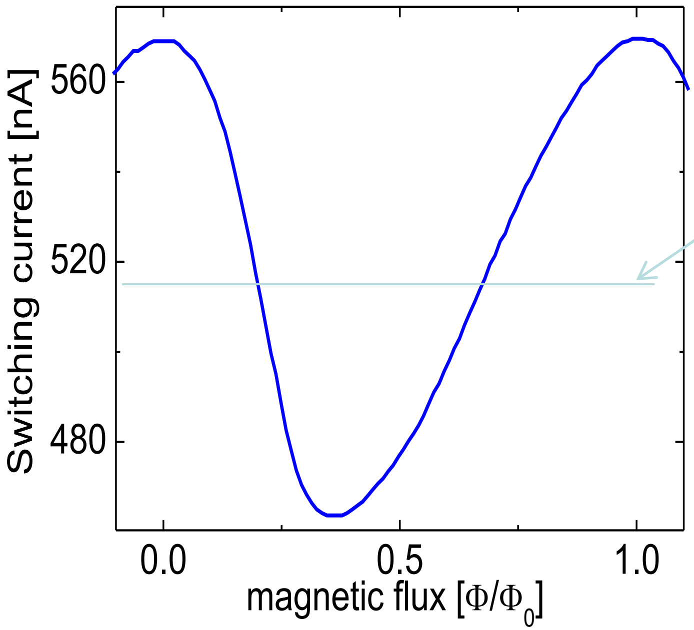




few channels, $\{\tau_i\}$ tunable



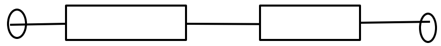
Interference pattern for ATOMIC SQUID



I0-switching current of junction alone

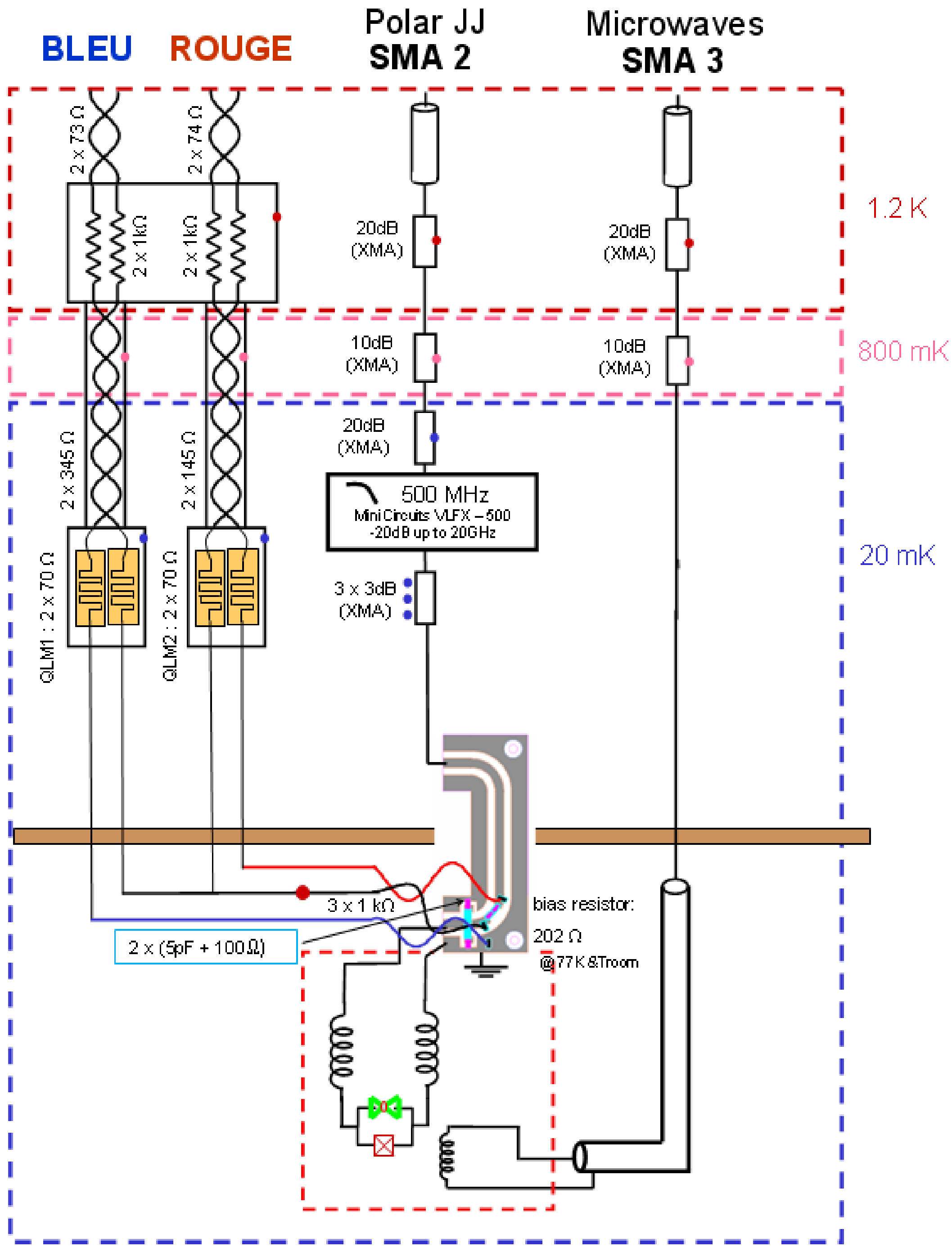
When SQUID switches, phase across JJ is the same independently of applied magnetic flux => interference pattern is current-phase relation of atomic contact

Voltage vs. phase

Normal metal	Superconductor
Voltage drop forces current	Phase drop imposes current
R r<<R  All voltage drops on R	The biggest phase drop in the loop on the weakest weak link

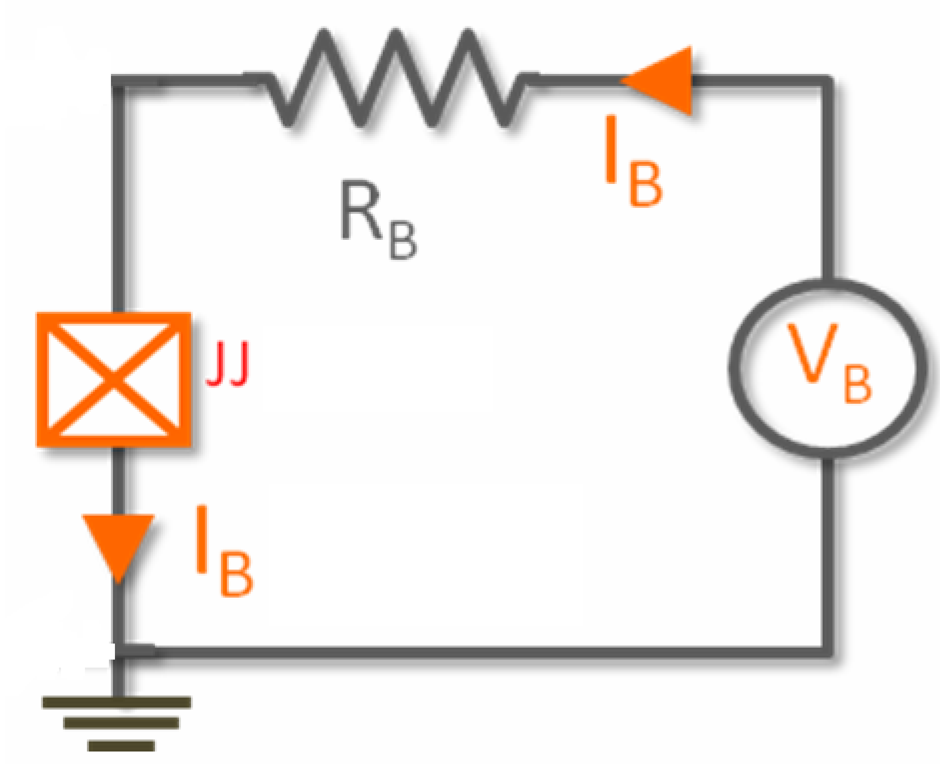
Wiring

of the fridge

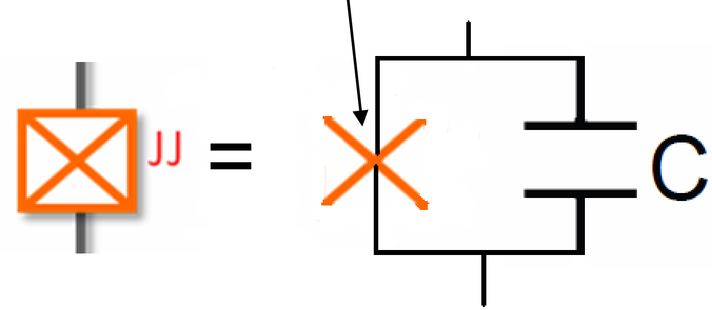


Electrical circuit

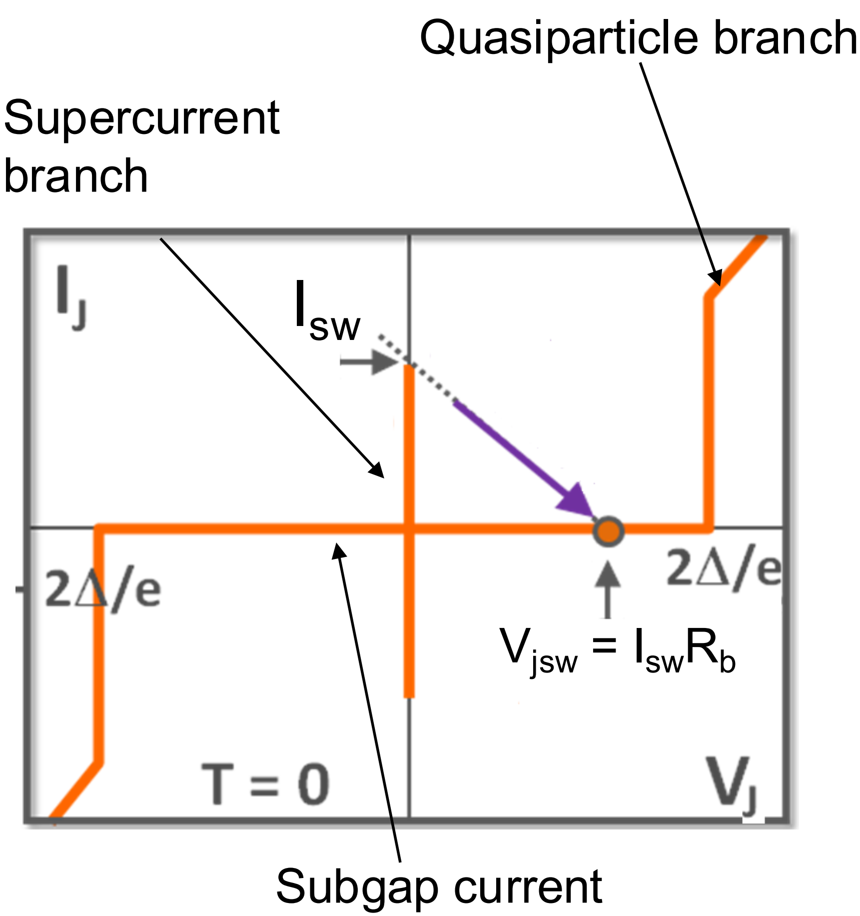
Thevenin equivalent



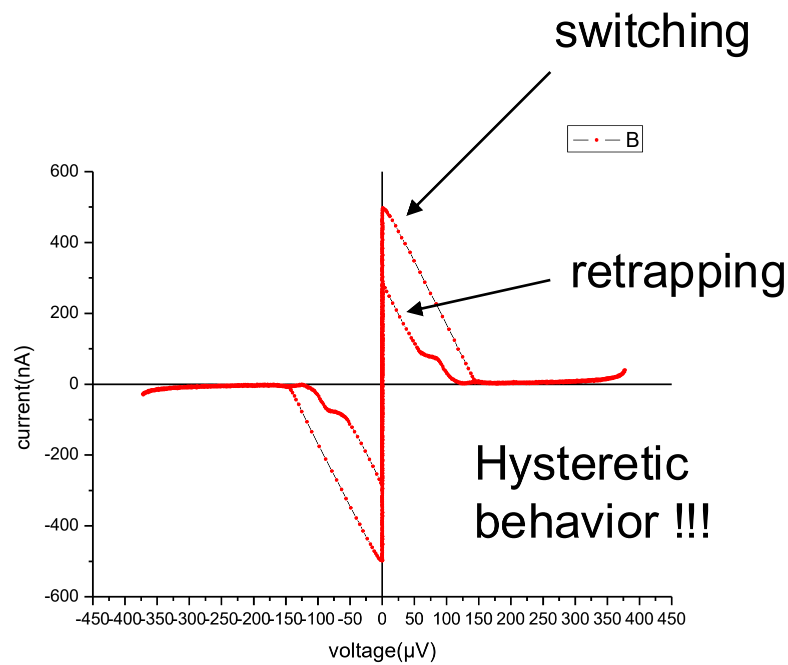
Pure Josephson element obeying Josephson relations



IV curve



I-V characteristics of JJ biased through R_B bias resistor. JJ supports supercurrent only to certain level. On crossing the threshold value I_0 finite voltage develops across JJ.



$$I_j = I_j(V_j)$$

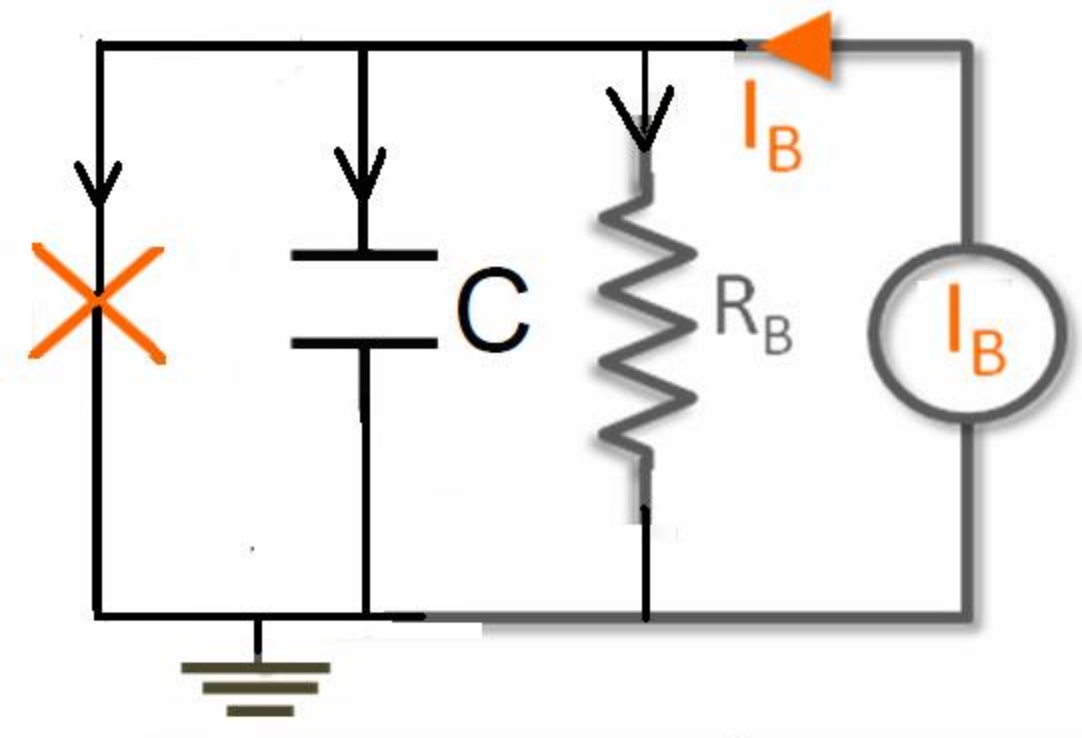
$$V_B = I_j \cdot R_b + V_j \Rightarrow I_j = -\frac{1}{R_b} \cdot V_j + \frac{1}{R_b} \cdot V_b \ (load\ \ line)$$

$$At\ switching\ V_j = 0, V_b = V_{bsw}\ and\ I_j = I_{sw} = \frac{V_{bsw}}{R_b}$$

$$I_j = -\frac{1}{R_b} \cdot V_j + I_{sw}\ (switching\ line)$$

RCSJ model

(Resistively and Capacitively Shunted Junction)



$$I_b = I_R + I_C + I_{JJ} = \frac{V}{R} + C\frac{dV}{dt} + I_0 \cdot \sin\gamma$$

RCSJ model

$$I_b = I_R + I_C + I_{JJ} = \frac{V}{R} + C\frac{dV}{dt} + I_0 \cdot \sin\gamma$$

$$\dot{\gamma} = \frac{1}{\varphi_0} \cdot V$$

$$I_b = \frac{\varphi_0}{R}\dot{\gamma} + C\varphi_0\ddot{\gamma} + I_0 \sin\gamma$$

Harmonic oscillator:

$$m\ddot{x} + b\dot{x} + kx = 0$$

$$E(t) = E_0 e^{-\gamma t}(energy\ of\ damped\ harmonic\ oscillator)$$

$$\gamma = \frac{b}{m}, \qquad Q = \frac{\omega_0}{\gamma}, \qquad -kx = -\nabla E_p$$

$$\ddot{x} + \frac{\omega_0}{Q}\dot{x} + \omega_0^{\ 2}x = 0$$

Q - quality factor, amplitude of harmonic oscillator falls by a factor of e in Q/π cycles of free oscillations

First, consider γ->0, I_B=0 and map it into harmonic oscillator:

$$C\varphi_0^{\ 2}\ddot{\gamma} + \frac{\varphi_0^{\ 2}}{R}\dot{\gamma} + \varphi_0 I_0 \gamma = 0$$

$$m = C\varphi_0^{\ 2}\ ,\ \omega_0 = \left(\frac{I_0}{C \cdot \varphi_0}\right)^{1/2}, b = \frac{\varphi_0^{\ 2}}{R}, Q_0 = \frac{\omega_0}{b/m} = RC\omega_0, k = \varphi_0 I_0 = E_J$$

Back to full equation:

$$\ddot{\gamma} + \frac{\omega_0}{Q_0}\dot{\gamma} + \omega_0^{\ 2}\left(\sin\gamma - \frac{I_b}{I_0}\right) = 0$$

$$\ddot{\gamma} + \frac{\omega_0}{Q_0}\dot{\gamma} + \omega_0{}^2(\sin\gamma - \frac{I_b}{I_0}) = 0$$

Wygląda jak harmonic oscillator, ale teraz restoring force wynosi nie kγ (jak w prawie Hooka) tylko:

$$F = -k(\sin\gamma - \frac{I_b}{I_0}) = -\nabla E_p$$

$$E_p = +k\int(\sin\gamma - \frac{I_b}{I_0})d\gamma = -E_J(\cos\gamma + \frac{I_b}{I_0}\gamma) \quad \textit{tilted washboard potential}$$

Tilted washboard potential

$$E_p = -E_J(\cos\gamma + \frac{I_b}{I_0}\gamma)$$

γ <-> x

V/φ_0 (napięcie) <-> v (prędkość)

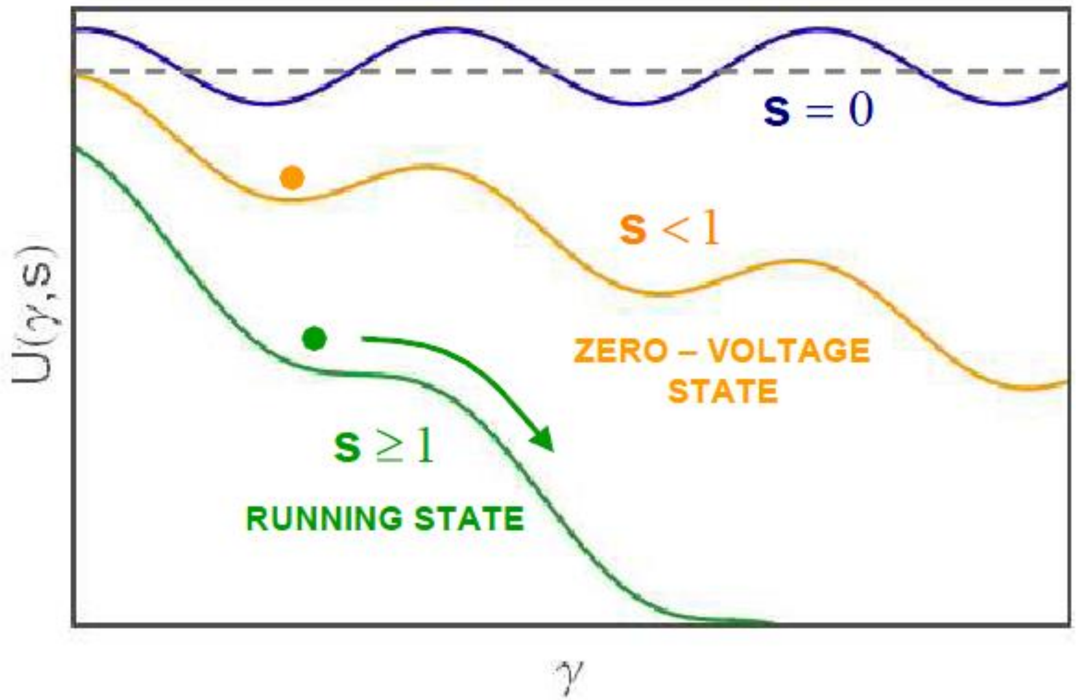


Fig. 2.3. Tilted washboard potential associated to the dynamics of the phase in absence of fluctuations (zero temperature in this classical model). For $s \leq 1$, the potential presents wells in which the particle is trapped. For $s \geq 1$, the wells disappear and the particle gets into to a running state.

Q (quality factor) <-> hysteresis

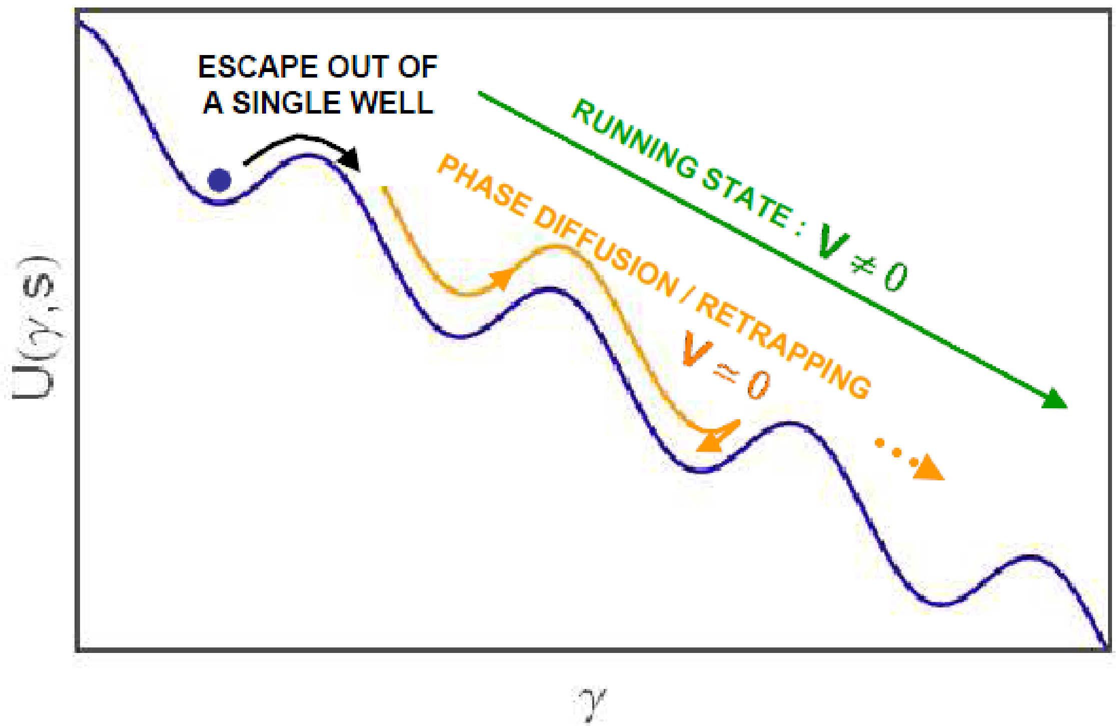


Fig. 2.4. Dynamics of the particle in presence of fluctuations (finite temperature in this classical model). The particle can overcome the barrier and escape out of the single well. The dynamics of the particle depends on the damping. If the damping is small, the particle gains enough energy to reach a running state, leading to a finite voltage. Otherwise, the particle enters a diffusion regime, escaping from one well to be retrapped in a following one, with a very small average velocity.