Summer school - Physics of Quantum Chips

June 30 – July 4, 2025 – University of Gdańsk, Gdańsk, Poland

Lectures, Peter Samuelsson, *Lund University*

1. **Quantum transport and tunneling in quantum dots.**
    - Landauer-Büttiker scattering approach to transport.
    - Double barrier structures and quantum dot tunneling.
    - Fermi's golden rule for tunnel rates.

Monday 30/6
9:00 – 10:30

2. **Density matrix description and quantum coherence.**
    - Density matrix – definition and properties.
    - Relaxation, dephasing and decoherence.
    - Quantum dot systems and charge coherence.

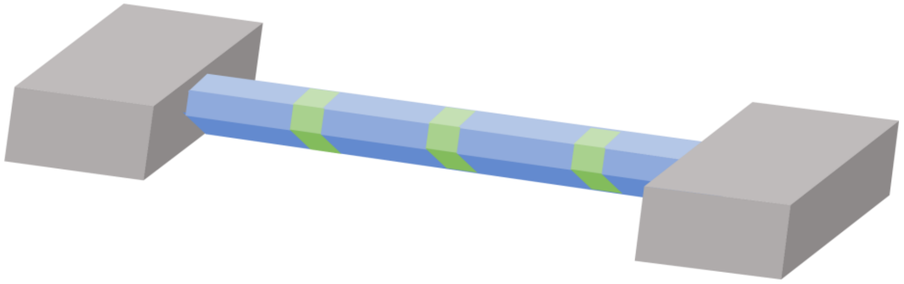
Tuesday 1/7
9:00 – 10:30

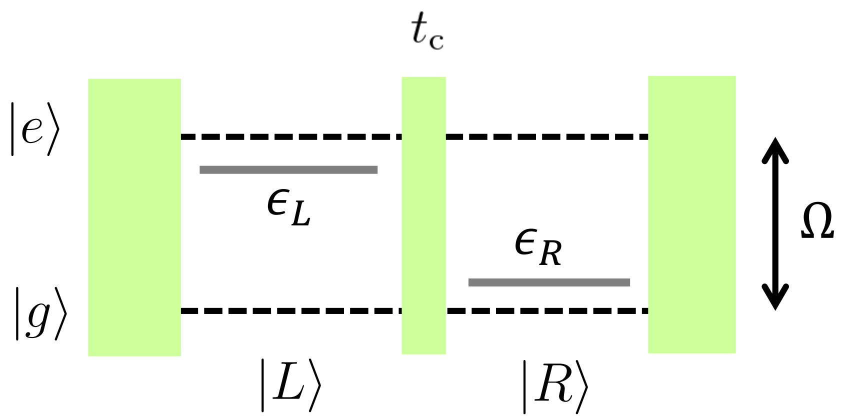
3. **Open quantum systems and Lindblad equation**
    - Lindblad equation, heuristic derivation.
    - Two-level system example.
    - Hybrid open quantum systems.

Wednesday 2/7
11:00 – 12:30

Quantum coherence – dots as qubits

Isolated double quantum dot (DQD) constitutes a charge qubit





Hamiltonian, local basis

$$H = \epsilon_L |L\rangle\langle L| + \epsilon_R |R\rangle\langle R| + t_c \left(|R\rangle\langle L| + |L\rangle\langle R|\right) = \frac{\epsilon_L - \epsilon_R}{2}\sigma_z + t_c \sigma_x + \text{const.}$$

Pauli matrices

Hamiltonian, energy eigenbasis

$$H = \frac{\Omega}{2}\left(|e\rangle\langle e| - |h\rangle\langle h|\right), \quad \Omega = \sqrt{(\epsilon_L - \epsilon_R)^2 + 4t_c^2}$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

Wavefunction, superposition

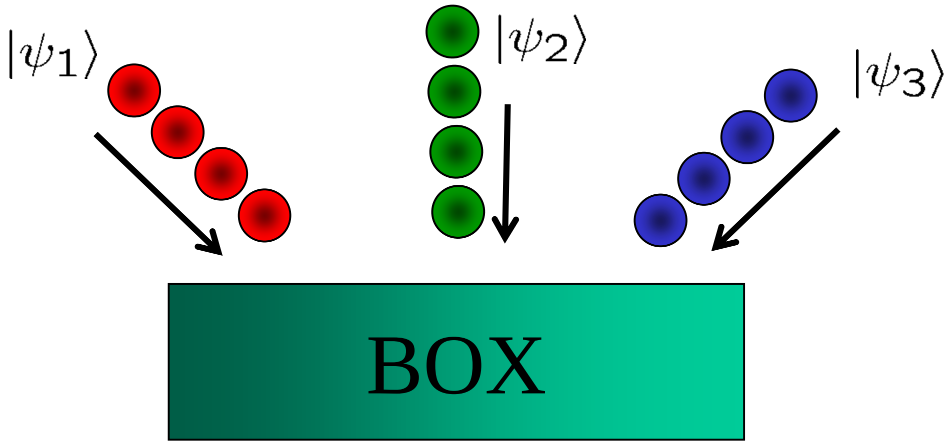
$$|\Psi\rangle = a|L\rangle + b|R\rangle = \alpha|e\rangle + \beta|g\rangle$$

Basis is important

How to handle loss of coherence, coupling to environment?

Density matrix/operator

Ensemble of pure states gives a mixed state



The density operator or density matrix ρ for the ensemble or mixture of normalized states $|\psi_i\rangle$ with probabilities p_i is given by

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \qquad \sum_i p_i = 1$$

Note: The set of states $|\psi_i\rangle$ and probabilites p_i are not unique:

$$\rho = \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1| = \frac{1}{2}\left[|a\rangle\langle a| + |b\rangle\langle b|\right] \qquad |a/b\rangle = \frac{1}{2}\left[\sqrt{3}|0\rangle \pm |1\rangle\right]$$

Fun facts about density matrices

- An operator ρ is a density matrix if (and only if):
  1) ρ has trace equal to one (probability cons.).
  2) ρ is positive semidefinite matrix (implies Hermiticity)

- A pure state $|\psi\rangle$ has a density matrix $\rho = |\psi\rangle\langle\psi|$.

- The purity $\frac{1}{d} \leq P \leq 1$ of a density matrix (d-dimensional space) is

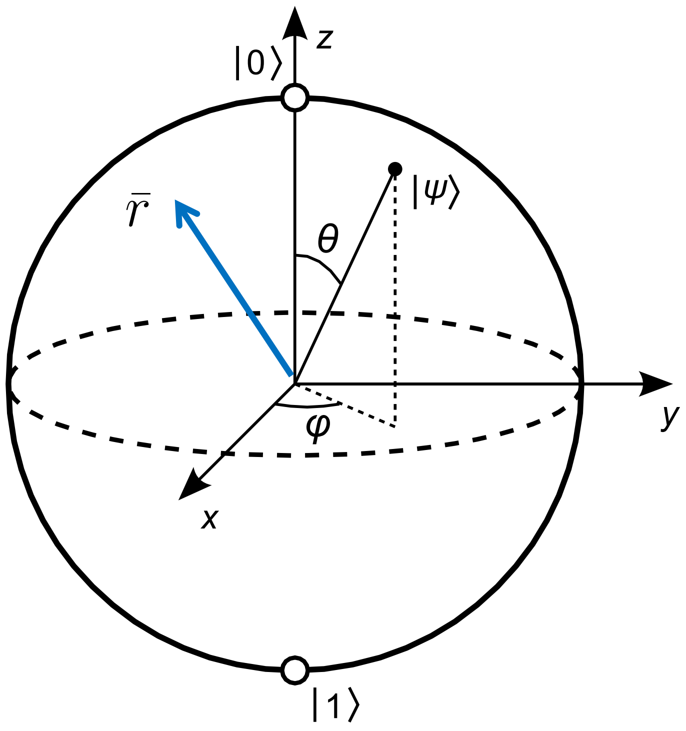
$$P = \mathrm{tr}\left(\rho^2\right)$$

- Any single qubit density matrix can be decomposed as

$$\rho = \frac{1}{2}[I + \bar{r} \cdot \bar{\sigma}] = \frac{1}{2}\begin{bmatrix} 1 + r_z & r_x - i r_y \\ r_x + i r_y & 1 - r_z \end{bmatrix}$$

where the Bloch vector $\bar{r} = [r_x, r_y, r_z]$, $\|\bar{r}\| \leq 1$
and $\bar{\sigma} = [\sigma_x, \sigma_y, \sigma_z]$.

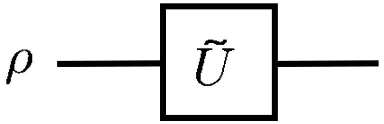
Bloch sphere



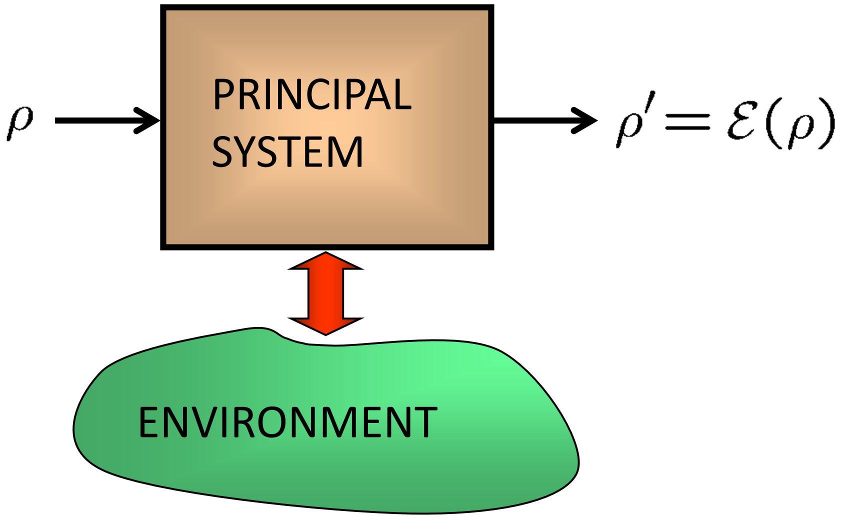
Loss of coherence of charge qubit: Purity, decomposition and Bloch vector.

Quantum operations

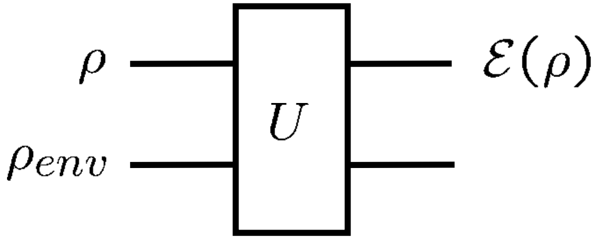
- Ideal, closed system



$\mathcal{E}(\rho) = \tilde{U}\rho\tilde{U}^\dagger$



- Non-ideal, open system



Total input state $\rho \otimes \rho_{env}$ rotation of system-environment U. Partial trace over environment

$$\mathcal{E}(\rho) = \mathrm{tr}_{env}\left[U(\rho \otimes \rho_{env})U^\dagger\right]$$

For no (principal) system-environment interaction $U = \tilde{U} \otimes I \Longrightarrow$

$$\mathcal{E}(\rho) = \tilde{U}\rho\tilde{U}^\dagger$$

Operator sum representation (POVM)

One can formulate the quantum operation $\mathcal{E}(\rho)$ in terms of operators acting on the principal system only

$$\rho' = \mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger,$$

an operator sum representation of $\mathcal{E}(\rho)$. The operation elements satisfy a completeness relation:

$$1 = \mathrm{tr}\,(\mathcal{E}(\rho)) = \mathrm{tr}\left(\sum_k E_k \rho E_k^\dagger\right) = \mathrm{tr}\left(\sum_k E_k^\dagger E_k \rho\right)$$

Since this holds for all ρ we must have (trace preserving)

$$\sum_k E_k^\dagger E_k = I$$

The set of operators $\{E_k\}$ is called a POVM (Positive Operator Valued Measure)

Note: The operator sum representation is not unique

$$\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger = \sum_k F_k \rho F_k^\dagger \qquad F_k = \sum_l U_{kl} E_l$$

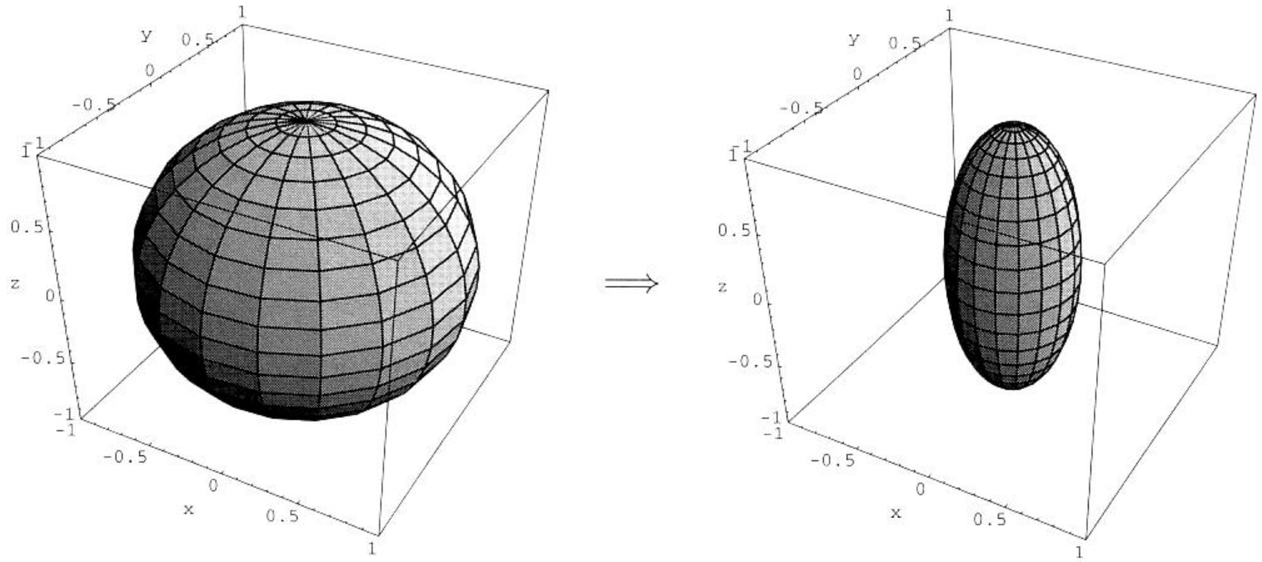
where U is a unitary matrix.

Loss of coherence of charge qubit: POVM

Illustrating qubit quantum operations

Dephasing: With probability $1-p$ the state is dephased.

$$E_0 = \sqrt{1-\eta/2}\, I = \sqrt{1-\eta/2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad E_1 = \sqrt{\eta/2}\, \sigma_z = \sqrt{\eta/2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$



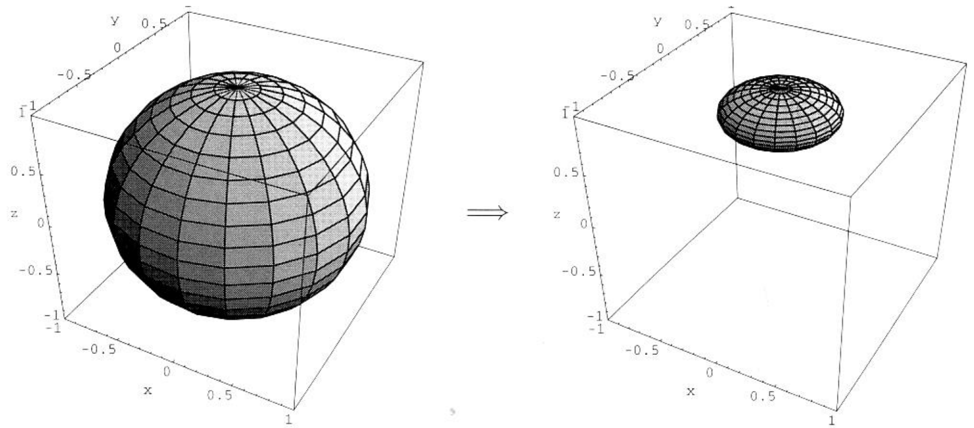
Depasing for $\eta = 0.6$. Left: Set of all pure states. Rigth: States after dephasing operation.

Nielsen and Chuang.

Relaxation: The system ”emits energy” with probability γ and approaches the ground state $|g\rangle$.

Operator elements (in the $|e\rangle, |g\rangle$ basis)

$$E_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{bmatrix} \qquad E_1 = \begin{bmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{bmatrix} \qquad E_0 = \begin{bmatrix} \sqrt{1-\gamma} & 0 \\ 0 & 1 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 0 & 0 \\ \sqrt{\gamma} & 0 \end{bmatrix}$$



Relaxation for $\gamma = 0.8$. Nielsen and Chuang.

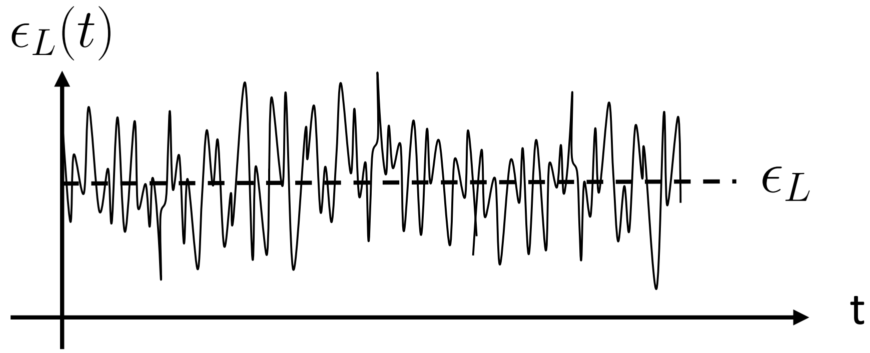
Relaxation of charge qubit

Dephasing and relaxation – environment fluctuations

DQD charge qubit under environmental fluctuations.

Hamiltonian parameters can acquire a part that fluctuates in time (often 1/f-noise)

$$\epsilon_{L/R}(t) = \epsilon_{L/R} + \delta\epsilon_{L/R}(t), t_c(t) = t_c + \delta t_c(t)$$



In general, $\delta\epsilon(t) = \delta\epsilon_L(t) - \delta\epsilon_R(t),$

$$H = \frac{\epsilon_L(t) - \epsilon_R(t)}{2}\sigma_z + t_c(t)\sigma_x = H_0 + \delta H(t),$$

$$H_0 = \frac{\epsilon_L - \epsilon_R}{2}\sigma_z + t_c\sigma_x, \;\; \delta H(t) = \frac{\delta\epsilon(t)}{2}\sigma_z + \delta t_c(t)\sigma_x$$

Convenient notation, $\bar{\sigma} = [\sigma_x, \sigma_y, \sigma_z]$,

$$H_0 = \bar{h}\cdot\bar{\sigma}, \quad h_x = t_c, h_y = 0, h_z = \frac{\epsilon_L - \epsilon_R}{2}$$

$$\delta H(t) = \delta\bar{h}(t)\cdot\bar{\sigma}, \;\; \delta h_x(t) = \delta t_c(t), \delta h_y(t) = 0, \delta h_z(t) = \delta\epsilon(t)/2$$

Statements

- For $\bar{h} \,||\, \delta\bar{h}(t)$ there is pure (only) dephasing.
- For $\bar{h} \nparallel \delta\bar{h}(t)$ there is both dephasing and relaxation.
- For $\bar{h} \perp \delta\bar{h}(t)$ there is minimum dephasing (sweet spot).

Calculate pure dephasing case

Paladino et al, Rev. Mod. Phys. 2014

Both relaxation and dephasing, example

The DQD case with energy level fluctuations $\delta\epsilon(t) = \xi(t)\Delta\epsilon$ ← Energy scale of fluctuations

$$H = H_0 + \delta H(t), \quad H_0 = \frac{\epsilon_L - \epsilon_R}{2}\sigma_z + t_c\sigma_x, \quad \delta H(t) = \frac{\delta\epsilon(t)}{2}\sigma_z$$

The fluctuations can now induce transitions between energy eigenstates $|e\rangle, |g\rangle$.

Under rather general conditions, starting at (t=0)

$$|\Psi(0)\rangle = \alpha|e\rangle + \beta|g\rangle$$

the density matrix becomes Paladino et al, Rev. Mod. Phys. 2014

$$\rho(t) = \begin{pmatrix} |\alpha|^2 e^{-t/T_1} & \alpha\beta^* e^{-t/T_2} \\ \alpha^*\beta e^{-t/T_2} & |\beta|^2 + |\alpha|^2\left(1 - e^{-t/T_1}\right) \end{pmatrix}$$

where

$$\frac{1}{T_1} = \pi\frac{t_c^2\Delta\epsilon^2}{\hbar^2\Omega^2}\bar{S}_\xi(\Omega) \qquad \frac{1}{T_2} = \frac{1}{2T_1} + \frac{1}{T_2^*} \qquad \frac{1}{T_2^*} = \pi\frac{(\epsilon_L - \epsilon_R)^2\Delta\epsilon^2}{4\hbar^2\Omega^2}\bar{S}_\xi(0)$$

Noise spectrum at energy splitting (→ $\bar{S}_\xi(\Omega)$)

Tuning to sweet spot removes "pure" dephasing part (→ $1/T_2^*$)